Characterizing Multiple Stationary Equilibria in Open Economies with Collateral Constraints*

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Abstract

This paper presents a new systematic approach to characterizing the *set* of stationary equilibria in the canonical small open economy model of sudden stops with price dependent collateral constraints. We first prove the existence of both sequential competitive equilibrium (SCE) and recursive competitive equilibrium (RCE). Relative to the set of RCE, we also provide a complete theory of equilibrium monotone comparative statics for all RCE relative to the deep parameters of the model, as well as provide a sufficient condition for the existence of a unique RCE. Then, using Generalized Markov equilibrium (GME) representations of SCE, we prove the existence of ergodic stationary equilibria. We show that the existence of an ergodic GME selectors depends critically on long-memory recursive representation of SCE (and, in particular, are not guaranteed to exist relative to RCE representations of SCE). We are also able to differentiate between short and long run global equilibrium stochastic dynamics and to find conditions that preserve the global stochastic stability of a suitable recursive equilibrium notion (i.e., an ergodic GME). For our results, the interplay between the RCE and the GME is critical. Finally, using numerical methods, we compare the properties of ergodic stationary equilibrium selectors vs other stationary equilibria (e.g., stationary equilibria as invariant measures).

Keywords: Financial Crises, Sudden Stops, Small Open Economies, Ergodicity, Recursive Equilibrium, Generalized Markov Equilibria

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1 Introduction

The recent literature on financial crises and sudden stops in emerging markets is voluminous.[1]. A critical element in this literature is the presence of an occasionally binding price-dependent equilibrium collateral constraint which is of central importance to generating emerging market financial crises through "pecuniary externalities" in this class of models. But it is also this feature that has also been argued to create the possibility of multiple sequential competitive equilibria (SCE). For example, in a recent paper by Schmitt-Grohé and Uribe ([74]), the authors show that pecuniary externalities can create multiple equilibria relative to SCE.[2]

The presence of multiple equilibria leads to a number of new important research questions from both a theoretical and applied perspective. First, dynamic equilibrium multiplicity complicates the characterization of stochastic equilibrium dynamics, and leads one naturally to ask if it is possible to provide a *unified* methodological approach to characterizing the *set* of possible SCE and/or recursive competitive equilibrium (RCE) dynamics as well compare their different equilibrium properties. Given the stochastic nature of these models, such a unified approach needs to be *global* (i.e., capable of constructing stationary equilibria from *any* initial state of the economy). It should also provide a sharp characterization of the sources of multiple dynamic equilibria. Additionally, it has been argued that the presence of the multiple SCE can generate "low borrowing" versus "high borrowing" dynamic equilibria (e.g., see Schmitt-Grohé and Uribe ([73], [74])), suggesting the *set* of dynamic equilibria possesses an *order theoretic* structure. This is also an important feature of the equilibria in this class of models that should be addressed. Finally, it would be interesting if one could develop tools that allow researchers to distinguish the stochastic equilibrium properties of different SCE and/or RCE selectors, which could possess differences in their short-run and long-run stochastic properties (e.g., differences relative to business cycle data in economies exposed to balance of payments crises).[3] Ideally, these tools should be accurate and numerically efficient.

This paper provides affirmative answers to all of these questions. In particular, the paper proposes a new constructive approach to the global characterization of the set of dynamic equilibrium in an infinite horizon general equilibrium stochastic economy with incomplete markets, where agents are subject to price dependent and occasionally binding constraints in the workhorse model in the small open economy literature (a two-sector endowment economy due to Bianchi ([15]) and Schmitt-Grohé and Uribe ([74], [75])). Our new tools provide a complete picture of how to handle the multiplicities inherent to this framework.

We first prove the existence of the SCE, RCE, and Generalized Markov equilibrium (GME) in these models. We then use these characterizations to prove the existence of an *ergodic* generalized Markov equilibrium. This latter result is surprising as, because of the presence of multiple equilibria, stochastic equilibrium dynamics are generally *discontinuous.* This latter result allows us then to explore the *interplay* between the minimal state space and the GME representations of SCE in the characterization of stochastic equilibrium dynamics, as well as derive a *computable* framework to approximate and characterize the long and short run properties of models with price-dependent inequality constraint and multiple equilibria.

[1] A small sampling of work in this literature includes the early work of Mendoza ([48], [49]), Mendoza and Smith ([50]), Biachi ([15]), Benigno, Chen, Otrok, Rebucci, and Young (e.g., see [11], [12]), Bianchi, Liu, and Mendoza ([16]). More recent papers include the papers of Schmitt-Grohé and Uribe ([73], [74], [75]), Arce, Bengui, and Bianchi ([6]), and Lutz and Zessner-Spitzenberg ([45]), Ottonello, Perez, and Vassaco ([59], [60]), Bengui and Bianchi ([14]), Rojas and Saffie ([66]), and Davis, Deverous, and Yu ([29]).

[2] It bears mentioning, the multiplicity result for SCE in Schmitt-Grohé and Uribe ([74]) pertains to the *deterministic* version of the model and is local to the steady-state. It is also important to note that we are not aware of *any* proof of existence of *any* SCE (or a recursive competitive equilibrium (RCE)) in either deterministic or stochastic versions of this model in the existing literature. So aside from developing new theoretical tools for characterizing the multiple stationary equilibria in these models, this paper also provides the first sufficient conditions for the existence of both SCE and RCE in the stochastic version the sudden stops model.

[3] Such differences in stochastic stability of SCE or RCE could also be critical when considering questions of econometric estimation (e.g., see the recent work on Benigno, Chen, Otrok, and Rebucci ([13])) for example) where conditions for the existence of *ergodic* stationary Markov equilibria could be very useful in the model's estimation.

1.1 Summary of Main Results

For the benchmark two sector small open economy with equilibrium price dependent collateral constraints that has been studied extensively in literature over the last decade, we first provide a general SCE existence result. We then sharpen our SCE existence and characterization results for the case of RCE, an important subclass of SCE that is a critical focal point of a great deal of numerical work in the recent literature. We believe our results of the existence of dynamic equilibria in this class of models are the first in the literature.[4] These existence results are not only of purely theoretical interest as they are necessary to guarantee our application of GME methods for characterizing multiple equilibrium stochastic dynamic. Such dynamics are well-defined *globally* from all admissible initial conditions of the stochastic model.

Relative to the set of RCE, we propose a new multistep monotone operator approach based on an equilibrium version of the Euler inequalities to characterizing the set of RCE. We prove that there exists a complete lattice of RCE. Further, we provide *explicit iterative tools* for computing the least and greatest RCE monotone comparative statics relative to deep parameters. Moreover, we also provide a generalized iterative procedures for computing monotone comparative statics of RCE/fixed point bounds from *any* RCE. In doing so, we are able to provide a *complete constructive qualitative theory* of *all* RCE for this class of models and connect them with the deep parameters of the model. In particular, relative to the literature that has focused on "low" and "high" borrowing dynamic equilibrium, we show that the least (resp., greatest) RCE correspond to "low" (resp., "high") borrowing RCE as mentioned in Schmitt-Grohé and Uribe ([74]). More to the point, we show how to compute each extremal RCE by simple successive approximations. In doing so, we also provide a new systematic approach based on monotonicity of our RCE operator that addresses the issue of "strict positivity" of equilibrium consumption in dynamic equilibria that has been discussed in the literature.[5] Finally, we provide the first sufficient conditions for the *uniqueness* of RCE in this class of stochastic models in the literature.

Our approach to constructing RCE is built on a new characterization of the structure of *implicit equilibrium complementarities* that exist between household and aggregate tradeables consumption. To obtain monotonicity for our RCE operator, we show that the well-known pecuniary externality that exists in these models when equilibrium collateral constraints bind in a RCE creates an *equilibrium single-crossing* condition between household tradeables consumption and per-capita aggregate levels of household tradeables that is parameterizing the relative price of nontradeables to tradeables in the equilibrium collateral constraint. For "low" vs. "high" borrowing RCE, for example, the central mechanism for equilibrium multiplicity is intuitive: if agents perceive the aggregate laws of motion of equilibrium states governing future per-capita aggregate borrowing will be "low" (resp., "high") in the future, RCE household collateral constraints will be tighter (resp., looser) because relative price of tradeables will be lower (resp., high) in a candidate perceived RCE, and these expectations are *globally self-generating* when the consumption aggregator that parameterizes relative price of tradeables generates multiple roots for implied maximal tradeable consumption via a version of the equilibrium collateral constraint. This, in turn, implies the existence of a least (resp., greatest) borrowing RCE where consumption/debt will be lower (resp., higher). We prove this *equilibrium* single-crossing condition provides a *global* source of *ordered* RCE.[6]

The paper then studies the set of stationary equilibria for these models. Here, using a novel version of a Generalized Markov Equilibrium (GME) approach, we are able to characterize and compare formally the stochastic equilibrium dynamics of *different classes* of SCE (including RCE as a subset). In particular, using GME methods, we construct representations of the stochastic dynamics derived from both the set of RCE as studied in Bianchi ([15]) as well as the set of sequential competitive equilibrium dynamics constructed in papers such as Schmitt-Grohé and Uribe ([73], [74], [75]) within the context of a single unified global methodological setting.[7] In doing so, we find some important differences in these classes

[4] Our tools appear to apply in more general classes of model with collateral constraints. (e.g., see Pierri and Reffett [64] for discussion).

[5] For example, see Schmitt-Grohé and Uribe ([74]) and Bianchi and Mendoza ([18]) for discussion of this issue.

[6] It is important to note, this single-crossing condition only appears in *equilibrium.* That is, household optimal decision for tradeables consumption are not monotone in per-capita aggregate levels of tradeables consumption.

[7] By "minimal state space recursive equilibria" (RE), we mean sequential equilibrium that admit a Markovian representation on a set of only minimal current state variables with no "endogenous" states. GME representations of sequential

of sequential stochastic equilibria. In particular, they have *very different* long-run stationary equilibrium properties. For example, by exploiting long-memory GME representations of SCE, we build a *global* theory of *ergodic* stationary equilibrium. That is, we prove the existence of such ergodic GME selectors, we show that structure of *memory* in the recursive GME representation of SCE plays a critical role in ergodic stochastic equilibrium dynamics. We also show that the existence of ergodic equilibrium selectors from the set of SCE *cannot* be guaranteed to exist if we restrict attention to only *short-memory* RCE selectors from the set of GME/SCE. As such short-memory RCE are the focus of much quantitative work in the literature (e.g., see Bianchi ([15]), for example), this results seems important.[8]

Our construction of ergodic stationary equilibrium is particularly surprising as it is well-known that the canonical sudden stops model can possess multiple sequential competitive equilibria, and hence stochastic equilibrium dynamics can easily become *discontinuous from initial conditions* . Therefore, characterizing SCE selections that induce ergodic equilibria of these models is very challenging endeavor.[9]

The key ingredients to identifying ergodic GME selections turns out to the occasionally binding equilibrium price-dependent collateral constraint and the flexibility of this equilibrium notion. As it is possible to embed some of the qualitative properties of the RCE into the GME, the equilibrium collateral constraint allows us to characterize the regeneration properties of stochastic equilibrium dynamics; a critical ingredient in proving the existence of ergodic GME selectors. In this sense, the *interplay* between the constructive and global characterization of the RCE and the flexibility of the GME is essential to achieve ergodicity. Mathematically, ergodic GME selectors are identified by constructing a so-called "atom" of the equilibrium Markov process. The process regenerates at the atom, a property that turns out to be deeply connected with *global stability* of stochastic equilibrium dynamics. In a sense, ergodicity of SCE selectors becomes a natural equilibrium selection device for characterizing short and long run stochastic dynamics among multiple SCE paths.

We also show GME methods can be used to compare stationary, non-stationary, and ergodic equilibrium selections (and their associated long-run stochastic properties). When doing this, using numerical methods, we find that ergodic selections have a different equilibrium stochastic structure that other GME selectors: they are less financially constrained and thus have smoother consumption paths. Even though the GME has a larger state space relative to the RCE, it can be computed efficiently. This is because we choose the additional state variable not only to expand the memory of the process but also to allow some paths of the GME to inherit the qualitative properties of the RCE. Thus, the interplay between these 2 equilibrium notions is also essential from a numerical perspective.

1.2 This Paper and the Existing Literature

This paper contributes to this literature in many ways in the context of the endowment two-sector small open economy model of Sudden Stops proposed in Bianchi ([15]).This model has been studied also extensively in a series of recent papers also by Schmitt-Grohé and Uribe ([73], [74], [75]), among others, where equilibrium multiplicity has been a focal point of the analysis. A common characteristic in the emerging market financial crises literature is the importance of equilibrium price-dependent collateral constraints in understanding the structure and genesis of the event. The collateral constraints are critical in explaining not only the mechanism by which financial crises occur by forcing emerging economies to pay down past debts, creating a possible collapse in consumption, prices, and current accounts, but also the observed co-movement in macro aggregates and the Fisherian deflation narrative suitable for emerging markets.

equilibrium critically require, among other things, additional state variable which are typically *endogenous.*

[8]RCE are "short memory" in the sense they are Markovian equilibrium that depend on only the current set of minimal states (and hence, 0-memory SCE). GME representations of SCE are recursive in general on the minimal state space only if one enlarges the set of state variables to include endogenous "states" such as envelope theorems, value functions, etc. For infinite horizon economies like those in this paper, GME representations of SCE could in principle have *infinite* memory when viewed from the perspective of a minimal state space.

[9]Although our focus in this paper is on GME methods and their ability to characterize stochastic equilibria is in the canonical model of sudden stops in the literature, as will be clear in the sequel, the central methodological theme of this paper is much more general, and makes the case for GME approaches in many stochastic equilibrium models with (endogenous) equilibrium collateral constraints and multiple equilibria.

Relative to the work of Bianchi ([15]), we extend his results in several directions. From a theoretical point of view, we provide a complete qualitative theory of how the *set* of RCE depend on the deep parameters of the model, as well as provide a new successive approximation/generalized "time-iteration" algorithm for computing least and greatest RCE as well as their equilibrium comparative statics. We also provide a new class of generalized iteration methods that can compute RCE bounds from any initial element of the domain of our RCE operator, as well as provide iterative methods for comparing *all* RCE as the parameters of the economy vary. The latter is particularly useful for numerical computation of RCE, as well as numerical comparisons of RCE under different parameter settings. Second, as in Bianchi ([15]), we study global stochastic equilibrium properties, but here we do this relative to the *set* of RCE. However, we also study stochastic equilibrium dynamics within the context of a GME representations (not just RCE representations) of SCE. The GME representations allow us to address many new and interesting features. For example, we can compare the stochastic properties of different types of stationary equilibrium within the context of the same model, and even deal with a non-stationary representation.

Relative to the work of Schmitt-Grohé and Uribe (e.g., [74], [75])), although the approach taken in this paper is very different than that taken in their work, our work complements their work along a number of new directions. First, exploiting the same source of equilibrium multiplicities discussed in their papers in the context of SCE, we globalize their results on existence of multiple equilibria for the case of RCE in the stochastic economy. Second, we take a very different quantitative approach to studying the properties of stochastic dynamic equilibria by using GME methods. In Schmitt-Grohé and Uribe ([74], [75]), their approach to characterizing the existence of multiple equilibrium is built upon the *deterministic* versions of the model, and in particular sequential equilibrium behavior "local" near a (deterministic) steady-state.[10] Our results are never "local" or "deterministic", rather always global and stochastic. We also show the existence of the pecuniary externality they focus on in their work implies an *implicit equilibrium complementarity* that generates the existence of *multiple stochastic stationary equilibria*. Our approach also builds a theory of stationary stochastic equilibrium from *any arbitrary initial condition,* providing a systematic approach to understanding the interplay between modeling multiple equilibrium, memory and the associated stationary equilibrium from vantage point of selectors from the set of GME representations of SCE. As in some of their work, what is critical in our approach to study stochastic properties of GME selections is the presence of "hit" times for the collateral constraint, and then use them to regenerate the equilibrium Markov process.

Our GME approach is general and can be potentially be applied in other models of Sudden Stops found in the existing literature such as models with elastic labor supply and production. Versions of the models with production include the early papers of Mendoza ([48]) and Mendoza and Smith ([50])), as well as the series of interesting papers by Benigno, Chen, Otrok, Rebucci, and Young (e.g., see [11], [12])).[11]

Our paper also is related to an emerging literature that seeks to characterize dynamic models with equilibrium borrowing constraints (and/or occasionally-binding constraints). Relative to this literature, we provide a new set of tools for characterizing the RCE and SCE in models with equilibrium price-dependent collateral constraints. Our multistep fixed point approach to RCE can be show to be useful on other related dynamic equilibrium models where stationary equilibrium partitions into states where collateral constraints are "slack" versus, "binding", and include models of credit cycles in the spirit of Kiyotaki and Moore ([39]),[12] models of financial frictions and production with collateral constraints such as Moll ([55]) or models of self-fulling credit cycles such as Azariadis, Kaas, and Wen ([7]).

This paper also contributes to the literature on self-generation methods and GME methods via strategic dynamic programming approaches to dynamic stochastic general equilibrium models (DSGEs). Self generation techniques were first introduced in the repeated games in the Abreu, Pearce, and Stacchetti ([1], [2]) and are related to implementations of the methods for studying the existence of Markovian equilibrium in dynamic stochastic models/games found in Blume ([20]) and Duffie, Geanokopolis, Mas-Colell, and McLennan ([31]). Approaches to making operational these methods in the context of GME represen-

[10]These papers then can characterize stochastic SCE dynamics using techniques building "stationary sunspot" approaches (e.g., see related work in Woodford ([82]) and Schmitt-Grohé ([70]), for example).

[11]These extensions are studied in Pierri and Reffett ([64]).

[12]See the survey of Gertler and Kiyotaki [35] for a nice discussion of this large literature.

tations and enlarge state spaces stems from the work of Kydland and Prescott ([42]), among others. [13]. A novel aspect of this paper is that by exploiting the structure of equilibrium price-dependent collateral constraints, we are able to propose a systematic approach to SCE selections, based upon ergodic GME selectors.

At least since the paper of Blume ([20]), when studying questions of equilibrium stochastic stability, it has along been recognized the trade off between the multiplicity of sequential equilibria and the continuity properties of the associated recursive representation. In other words, the presence of multiple equilibrium can generate discontinuities if dynamic equilibria over minimal state space equilibrium transition functions. In some cases, by enlarging the state space, it is possible to obtain a continuous Markov equilibrium (see Pierri, [62]); but unfortunately, there is no general theory about how to do this (see Kubler and Schmedders, ([40]), for a counter-example). Interestingly, in the context of models with price dependent collateral constraints, we show that the GME methods we develop in section 4 are critical in the presence of this type of discontinuities, which may affect the existence of an ergodic equilibrium.

Our work is also directly related to the large literature on the equilibrium comparative statics in dynamic economies using monotone-map methods (or "time-iteration") methods.[14] In a recent paper, Datta, Reffett, and Wozny ([27]) propose a multistep monotone-map/time iteration method that proves especially suited for dynamic models with multiple RCE. Our paper extends the class of multi-step monotone-map methods to dynamic models with equilibrium price-dependent collateral constraints. In addition, our paper is also related to Mirman, Morand, and Reffett ([54]) and Acemoglu and Jensen ([3]) as it provides sufficient conditions for monotone dynamic equilibrium comparative statics in the deep parameters of the economy. In particular, this paper extends these results into the dynamic models with equilibrium price-dependent borrowing constraints.

Given the approach to characterizing dynamic equilibria in this paper, we also address many of the interesting questions raised in recent work that discussed the critical difference between local versus global methods for these models relative to solving equilibrium functional equations (e.g., as discussed in the new work of De Groot, Durdu and Mendoza ([30]) and Mendoza and Villallazo ([51])), but also focuses on the implications of GME methods for characterizing, computing and simulating ergodic, stationary and non-stationary equilibria.

The remainder of the paper is as follows: in section 2 discusses the model, and proves the existence of SCE. In section 3, we characterize the set of RCE for these models. Section 4 then contains the results for the GME and section 5 a) discusses the interplay between the SCE, the RCE and the GME, and how to handle multiple equilibrium from a constructive perspective, b) characterizes multiple equilibrium in a stochastic setting, c) solves the model, compute and simulate ergodic, stationary and non-stationary equilibrium. Section 6 concludes with a few remarks about further research questions and extensions.

2 The Model and Sequential Competitive Equilibrium

We consider the endowment version of the two-sector small open economy model studied in Bianchi ([15]) and Schmitt-Grohe and Uribe ([73], [74], [75]). The model is a small open economy with a fixed interest rate. Time is discrete over an infinite horizon and indexed by $t \in \{0, 1, 2, ...\}$. There is a representative agent and two sectors of perishable goods, a tradable consumption good y_t^T and a non-tradable consumption good y_t^N. Each household is endowed a strictly positive amount of each good in each period. Upon receiving their current period endowments, households sell endowments at current market prices and choose consumption of both goods. The consumption of tradeable and non-tradeable is denoted, respectively, by c_t^T and c_t^N. It turns out to be useful to take as the numeraire the tradable good, so the relative price of non-tradeable relative to the numeraire tradeable in period t is denoted by p_t.

Household preferences are defined over infinite sequences of dated consumption vectors of tradeable and non-tradeable goods $c_t = (c_t^T, c_t^N) \in X \supset \mathbf{R}_+^2$ where X is the commodity space for consumption of tradeable and non-tradeable in each period, and are assumed to be time separable with subjective discount

[13]See Kubler and Schmedders ([41]), Feng, Miao, Peralta-Alva, and Santos ([33]), and Cao ([22]).

[14]This literature started with the papers of Coleman ([23], [24]), and was extended in Datta, Mirman, and Reffett ([26]), Morand and Reffett ([56]), and Mirman, Morand, and Reffett ([54]), among many others.

factor $\beta \in (0,1)$. These preferences are represented by a nested utility function which is a composition of two functions: a utility over composite consumption $U : \mathbf{R} \longrightarrow \mathbf{R}$, and an aggregator $A : X \longrightarrow \mathbf{R}$ over tradeable and non-tradeable consumption $c_t = (c_t^T, c_t^N)$, where the preferences $U(A(c))$ gives the instantaneous utility of the vector of consumption $c \in X$ in any period. Then, lifetime discounted expected utility preferences of a typical household are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{1}$$

where the mathematical expectation operator here is taken over the stochastic structure of uncertainty with respect to the date 0 information.[15]

Uncertainty in the economy is modeled as an iid stochastic process governing tradeable endowments $y = \{y_t^T\}_t$, where each element of sequence y has distribution given by the measure $\chi(\cdot)$. Here, for convenience, we assume the sequence of non-tradeable endowments $\{y_t^N\}_t$ is non-stochastic as it plays no role in the characterization of stochastic equilibrium dynamics in this paper. We assume further that the realizations for tradeable endowments in any period denoted by $y_t^T \in Y$ where the shock space Y is finite set.[16] By an application of standard results, these assumptions on shocks imply it is possible to define a stochastic process $(Y_\infty, \Omega, \mu_{y_0^T})$ which takes realizations in each period in Y. [17] Given this fact, denote by Y_∞ the space of infinite sequences in Y, and assume $y_0^T \in Y$ is the initial condition for this stochastic process for tradeable.

Households face a sequence of budgets constraints when making their sequential choices for consumption and debt over their lifetimes. In particular, given a candidate price sequence $p = \{p_t\}_{t=0}^{\infty}$, and denoting the net debt position for a typical household with debt borrowed at date t but maturing at date $t+1$ by d_{t+1}, the budget constraint for a household in any period t is given by:

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R} \tag{2}$$

where agents are allowed to borrow or lend at a fixed interest rate $R = 1 + r$ and, as this is a small open economy, R is taken as given. The sequential budget constraints here follow the timing convention used in Schmitt-Grohé and Uribe ([74], [75]), and assume consumption and income decisions are taken at the beginning of the period, and interest is then paid/earned over that same period. We adopt this timing only because it proves to be convenient in characterizing the structure of dynamic equilibrium.[18] In addition to the budget constraint in (2), a typical household also faces a period by period flow collateral constraint on debt given by:

$$d_{t+1} \leq \kappa(y_t^T + p_t y_t^N) \tag{3}$$

where $\kappa > 0$ [19]

The collateral constraint in (3) is an occasionally binding price-dependent collateral constraint that is *endogenous* and an *equilibrium object.* In fact, we shall prove that in equilibrium this endogenous

[15]A typical functional form for the consumption aggregator$A(c)$ in the literature is the Armington/CES aggregator

$$c_t = A(c_t^T, c_t^N) = [a c_t^{T\,1-\frac{1}{\xi}} + (1-a) c_t^{N\,1-\frac{1}{\xi}}]^{\frac{1}{1-\frac{1}{\xi}}}$$

with $\xi > 0$, $a \in (0,1)$, which is increasing, strictly concave, and supermodular on X when $X = \mathbf{R}_+^2$ with its product order.

[16]All the results of sections 2 and 3 on the existence on SCE and RCE hold for more general shocks (i.e., endowment processes for tradeables consumption that follow a first order Markov process with stationary transition $\chi(y^T, y^{T\prime})$ on a continuous shock space but with substantial differences in proofs (especially for the case of SCE). See Pierri and Reffett ([64]).

[17]e.g., see Stokey, Lucas and Prescott ([78], chapter 7)).

[18]Bianchi ([15]) uses a slightly different timing convention, but it turns out this timing convention is without loss of generality in our case (see, for example, Adda and Cooper ([4]) for a detailed discussion of this matter).

Also, in this paper, we do not address the important question of "future" vs "current" wealth collateral constraints as discussed in Brooks and Dovis ([19]) and Ottonello et al ([60]). Some cases of "future" wealth collateral constraints can be put into our framework as discussed in Pierri and Reffett ([64]).

[19]As is well-known in this literature, one can write down more fundamental versions of this model where this debt constraint emerges as an equilibrium object from the primitives of the underlying economy. For our purposes, we just follow the literature and impose this form of a price-dependent collateral constraint.

constraint is the source of equilibrium complementarities (and hence, equilibrium multiplicities) in these models. As mentioned earlier, we shall show the endogeneity of the equilibrium collateral constraint will create a type of equilibrium *pecuniary complementarity* into this model. In particular, when agents believe in equilibrium paths for per-capita tradeable consumption will be "higher", the equilibrium collateral constraint is relaxed, allowing household tradeable consumption to expand via access to additional debt and tradeable consumption. In a RCE, for example, we shall show that the equilibrium collateral constraints are themselves *ordered.*

2.1 Existence of Sequential Competitive Equilibrium (SCE)

We now consider sufficient conditions for the existence of a SCE. In a SCE, the representative household takes as parametric an interest rate R, a level of initial debt $d_0 \in D$ (where $D \subset \mathbf{R}$ is a compact set of debt states which will be constructed in a moment), a stochastic process governing $y^T = \{y_t^T\}_{t=0}^{\infty}$ conditional on an initial level of tradeable endowment $y_0^T \in Y$ with $y_0^T > 0$, a constant level of non-tradeable endowment $y_t^N = y^N$, and a history-dependent sequence of measurable prices $p = \{p_t(y^t)\}_{t=0}^{\infty}$, where we use the notation $y^t = \{y_0^T, y_1^T, .., y_t^T\}$ to denote the history of tradeable endowment realizations up to period t, and it maximizes lifetime utility in (1) subject to constraints (2) and (3) for each time period. That is, given $R \geq 0$, d_0, the stochastic processes p and y, the household solves the following:

$$V^*(s_0, p, R) = \max E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{4}$$

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R}; \quad d_{t+1} \leq \kappa(y_t^T + p_t y_t^N),\ t \in \{0, 1, 2, ...\} \tag{5}$$

where the initial states are $s_0 = (d_0, y_0^T)$, and $y_0^T \in Y$. We denote the optimal policy sequences for consumption and debt achieving the maximum on (4) by

$$c^*(s_0, p, R) = \{c_t^*(s_0, p, R)\}_{t=0}^{\infty};\ d^*(s_0, p, R) = \{d_t^*(s_0, p, R)\}_{t=0}^{\infty}\} \tag{6}$$

where in a moment we shall impose sufficient convexity and continuity conditions on the primitive data of the model such that (a) the value function $V^*(s_0, p, R)$ is finite, and (b) optimal sequences $c^*(s_0, p, R)$ and $d^*(s_0, p, R)$ are well-defined and unique.

Further, when in a moment we impose Assumptions 1 on preferences, the households sequential optimization problem will satisfy standard convexity, continuity conditions, and continuous differentiability assumptions on preferences, and by an appeal to well-known duality arguments in the literature (e.g., Rincon-Zapatero and Santos ([65]), theorem 3.1) , we can show that there exists a well-defined standard Lagrangian formulation for the sequential primal problem in (4) with (summable) dual variables $\beta^t \lambda_t$ and $\beta^t \lambda_t \mu_t$ associated with the sequence of constraints (2) and (3), respectively, and well as standard envelope theorems from date 0 household states. Noting our constrained system satisfies sequential linear independence constraint qualifications, strong duality holds between the resulting Lagrangian formulation and the primal program in (4) and the infinite dimensional system of KKT multipliers are well-defined and unique. We can then formulate the system of first order conditions for this problem in a sequential competitive equilibrium using the Lagrangian dual as follows: the optimal stochastic processes $c^*(s_0, p, R)$ and $d^*(s_0, p, R)$ satisfy

$$\lambda_t^* = U'(A(c_t^*))A_1(c_t^*) \tag{7}$$

$$p_t = \frac{A_2(c_t^*)}{A_1(c_t^*)} \tag{8}$$

$$[\frac{1}{R} - \mu_t^*]\lambda_t^* = \beta E_t \lambda_{t+1}^* \tag{9}$$

$$\mu_t^*[d_{t+1}^* - \kappa(y_t^T + p_t y_t^N)] = 0, \mu_t^* \geq 0 \tag{10}$$

Equations (7) to (10) characterize optimization. Before defining an equilibrium, we must describe feasibility. As it is standard in the literature, we only require the market of non-tradable goods to clear: $c_t^N = y^N$. Then, a SCE for this economy is then defined as follows: [20]

Definition 1 *A Sequential Competitive equilibrium (SCE) is a collection of progressively Ω-measurable random variables for consumption $c^*(s_0, p^*(s_0, R), R)$, debt $d^*(s_0, p^*(s_0, R), R)$, and relative prices of non-tradeable to tradeable consumption $p^*(s_0, R)$ such that: 1) the representative agent chooses $c^*(s_0, p^*, R)$, $d^*(s_0, p^*, R)$ to solve (4) given s_0 at $p^*(s_0, R)$ such that $V^*(s_0, p, R)$ is finite and equations (7-10) hold, and 2) markets clear $c_t^{N*}(s^t) = y^N$ where y^t holds $\mu_{y_0^T}$-a.e*

Before discussing our sufficient conditions for the existence of SCE, we need to mention a few key technical issues that arise when studying the structure of SCE in stochastic infinite horizon debt models. The first issue is to place enough structure on the model's primitive data so that we can characterize the (nonempty) set of *stochastic steady state* distributions for the model (i.e., the set of *stationary equilibrium).* A stationary equilibrium in this model can be associated with either the SCE and/or the recursive equilibrium of the model. So we need to impose sufficient structure on the model such that we can obtain sufficient boundedness (actually, compactness) for the stochastic equilibrium dynamics of the model. In the existing literature, the question of how to provide lower and upper bounds for stochastic equilibrium paths for endogenous variables in small open economy frameworks such as ours has not been addressed. Yet, it is well known that in stochastic models of debt, issues related to lower and upper bounds can be delicate.

To impose sufficient structure to obtain such lower and upper bounds on stochastic sequential equilibrium dynamics, what is typically done in the existing literature to impose bounds on the marginal utility of consumption (in addition to an otherwise standard conditions on the model's primitive data). These bounds on marginal utilities involve modifying standard preferences in the literature in a manner we will define in a moment when we state our sufficient conditions for existence of SCE.[21] What is critical to note here is that these type of restrictions *do not affect the stochastic dynamic behavior* of the model when compared to the more "traditional" CES preferences (as we shall discuss in the quantitative section of the paper in section 5). [22] They are critical though to insure the necessary compactness of the sequential equilibrium set, as well as provide the basis for the existence of a stationary and compact state space in both the SCE and recursive (Markovian) formulations of the dynamic equilibrium of the model.[23] So first we show that any sequential competitive equilibrium in this economy is compact.

We then state our assumptions on the primitive data of the model as follows:

Assumption 1: The functions $U(x)$ and $x = A(c)$ satisfy the following conditions: a.i) $lim_{x \longrightarrow \infty} U'(x) = 0$ or *a.ii) $\exists x \in X$ such that $\forall y \in B_\epsilon(x) \quad U(y) \leq U(x) \quad \epsilon > 0$,* a.iii) Let $A(.,.)$ be a function mapping $X \mapsto \mathbb{R}_+$, where X is the consumption space, b) $\mathbb{R}_+^2 \subset X, X$ open, c) $A_1(c) = A_1(., c_2) \in \mathbb{R}$, for all $c_2 \in X \mid_{c_2}$, $A_2(c) = A_2(c_1, .) \in \mathbb{R}$ for all $c_1 \in X \mid_{c_1}$, where $X \mid_{c_j}$ is the projection of the consumption space on its c_j component for $j = 1, 2$, [24] d) $A_2(c)/A_1(c) = A_2(c_2)/A_1(c_1)$ and $A_2(c_2)/A_1(c_1)$ is increasing in c_1 given $c_2 = y^N$, e) $lim_{c_1 \longrightarrow \infty} A_1(c_1, c_2) = cl_1 > 0$ for all $c_2 \in X \mid_{c_2}$, f) $lim_{c_1 \longrightarrow 0} A_1(c_1, c_2) = cu_1 < \infty$

[20] In the appendix, where we prove existence of sequential competitive equilibrium, we had the formalities of measurability of SCE processes more rigorous. See the details there.

[21] We should mention, when we prove the existence of RCE (a type of SCE), we use completely standard conditions on utility $U(A(c))$ that allow for standard Inada conditions near zero consumption. Such a condition is very helpful (although not necessary) to prove the existence of strictly positive RCE tradeable consumption. Proving the existence of RCE under just Assumption 1 can also be done also, but constructing the least RCE with strictly positive consumption is not clear. The proof of the greatest RCE, though, with strictly positive consumption is straight-forward. See Pierri and Reffett ([64]) for added discussion.

[22] That is, when studying the quantitative stochastic properties of these models in calibrated settings, compactness of the state space if often present without these types of "boundary" conditions on preferences

[23] The compactness needed is in the sense that realizations of equilibrium random variables are all contained in a compact subset of a finite dimensional space.

[24] This assumption is made to guarantee that the limit is well defined as we need to impose an upper / lower bound on the derivative for the limits. Also, the notation A_j denotes the partial derivative with respect to the $j = 1, 2$ argument of A.

for all $c_2 \in X \mid_{c_2}$, g) $U(A(c)) : X \longrightarrow \mathbb{R}$ *is* C^1 *continuous, strictly increasing, supermodular, and strictly concave.*

We make a few additional remarks on these assumptions.

Assumption 1.a.ii can be used to prove the existence of any SCE only in the case where $\beta R \geq 1$, which is typically missing in the literature. It simply says that lifetime preferences must have an asymptotic "satiation point" $x \in X$. As these preference structure affects the underlying stochastic process significantly, attracting equilibrium paths to the satiation point, we defer the treatment of this case to future research. Thus, we focus in the traditional case $\beta R < 1$. That is, we can dispense with Assumption 1.a.ii if the discount factor is sufficiently low.

Assumption 1.a.iii gives us some flexibility in the choice of A as it is standard in the literature. Assumption 1.b, 1.c, 1.e, and 1.f insures sequential equilibrium prices are bounded above and bounded below away from zero. This, in turn, will imply that in a SCE, debt is bounded above due to the collateral constraint in (3) and bounded below as wealth will be finite at every possible node. The online appendix contains examples of aggregators satisfying these restrictions (see the supplementary material for section 2).

It is possible to relax Assumption 1.e and 1.f by restricting the superdifferential, the set which includes all possible supergradients, of the concave mapping c on X to be compact. However, in applications, preferences are assumed to be stronger than Assumption 1.g and continuously differentiable. In this case, the strengthening of Assumption 1.e and 1.f can be done without loss of generality.

Also, when studying the existence of the RCE and of the GME, we can replace Assumption 1.f with the assumption that preferences satisfy an standard Inada condition. This assumption imposes, in effect, and Inada condition on $A(c)$ in tradeables consumption. The point is when constructing RCE, the characterization of stochastic equilibrium dynamics can be greatly sharpened (in particular, its uniform interiority of tradeable consumption) allowing us to characterize the (strong) interiority properties of stochastic dynamic equilibrium consumption paths. The complications comes in studying the SCE (and in particular, the compactness of the stochastic dynamics). For this, we need Assumption 1 as stated. The following remark formalizes this last paragraph. [25]

Remark 2 *For the RCE in section 3 and for applications in section 5 assumptions 1.b), 1.e) and 1.f) can be replaced by assuming that $U(x)$ is unbounded below and bounded above and that debt d has a uniform upper bound.*

Assumption 1.g is standard in the general equilibrium literature (see for instance Braido ([21])) . In applications, Assumption 1.g will be insured by assuming that the set X is uniformly bounded below, $U : \mathbf{R} \longrightarrow \mathbf{R}$ and $A : X \longrightarrow \mathbf{R}$ are both increasing, concave, and twice-continuously differentiable, and $A(c)$ additionally supermodular.

To define the recursive equilibrium notions in this paper, we will characterize optimal decision in primal form. That is, first order conditions are given by:

$$[c^N(y^t)][\{\beta^t U'(A(c(y^t))\}\{-A_1(c^T(y^t))p(y^t) + A_2(c^N(y^t))\}] = 0, \quad y^t - a.e \tag{11}$$

$$\begin{gathered}[\kappa\{y^t + p(y^t)y^N\} - d(y^t)][U'(A(c(y^t))A_1(c^T(y^t)) \\ -E_t(U'(A(c(y^t y_{t+1}))A_1(c^T(y^t y_{t+1}))] = 0, \quad y^t - a.e\end{gathered} \tag{12}$$

[25]Standard CRRA function for U with a CRRA parameter bigger than 1 insures the existence of a uniform lower bound on aggregate consumption A. The monotonicity of $U(A(c))$ and the fact that X is bounded below by zero, implies that c^T, c^N are both uniformly bounded below by zero; a fact that replaces assumption 1.f. The upper bound on d implies that consumption is bounded above as assets has a uniformly bounded above under CRRA preferences (see lemma 1 in Braido ([21])). As consumption is bounded above we can dispense with 1.e). Moreover, with these preferences the stationary system of equations formed by equations (8) and (10) has a finite number of roots for any finite d, Y^T. By setting the upper bound of d equal to the maximal root (across d, y^T), we can prove the existence of an ergodic equilibruim using the arguments in section 4.

We now state our first critical result for constructing the existence of SCE: [26]

Lemma 3 *Suppose $\beta R < 1$. Under assumptions 1-a.i, 1-a.iii, 1-b to 1-g,* ***if*** *a SCE (c, d, p) exists, then: (a) [Compactness provided existence] $[c(y^t), d(y^t), p(y^t)] \in K_1 \times K_2 \times K_3 \subset \Re^4 \quad y^t - a.e.$ and uniformly in $[y_0, d_0] \in Y \times K_2$, where $K_1 \times K_2 \times K_3$ is compact, (b) [Necessity of first order conditions] the SCE must satisfy* (11),(12) .[27]

We mentioned that following the results in Rincon-Zapatero and Santos ([65]), Lemma 3 gives an equivalent characterization of the SCE in terms of the equations (7)-(10) with well defined sequential KKT. Also, notice that this lemma does not guarantee the existence of the SCE, which will be considered in Theorem 4 below.

We also should mention, if we allow the cardinality of Y to be arbitrarily large, proving the existence of this type of equilibria can be rather challenging (See, for instance, Mas-Collel and Zame, [46]). Once the almost every where compactness of any "suitable candidate for equilibria" (c, d, p) is proved using lemma 3, the existence of the SCE will be proved assuming that Y is a finite set. MasColell and Zame ([46]) needed to state by assumption the uniform compactness of the SCE for the case of uncountable shocks. As it is natural, to show the existence of a recursive representation in this last case, we must impose the same assumption.

Finally, we give sufficient conditions for the existence of the SCE.

Theorem 4 *Suppose $\beta R < 1$ and the Y is a finite set. Under Assumptions 1-a.i,1-a.iii, 1-b -1-g, then: (a) [Existence] there exist a SCE, (b) [Sufficiency of first order conditions] any $[c(y^t), d(y^t), p(y^t)]$ that satisfy* (11),(12) *and $c_t^N(s^t) = y^N$ where y^t holds $\mu_{y_0^T}$-a.e is a SCE.* [28]

3 Characterization of Recursive Competitive Equilibrium

We now consider the existence of RCE for this class of models. We begin by relaxing Assumption 1.f to allow for a standard Inada condition when considering the case of RCE. This, along with Assumption 1.e facilitates the proof that the implied SCE prices $p = \{p\{S_t\}\}$ associated with any RCE will be bounded away from 0 with strictly positive RCE tradeables consumption.[29]

Assumption 2. *$U(A(c^T, c^N))$ satisfies the Inada condition for tradeables consumption: for $c = (c^T, c^N)$, $c^{NT} > 0$, $\lim_{c^T \to 0} U'(A(c))A_1(c) \to \infty$.*[30]

To construct RCE, we begin by representing the aggregate economy recursively on a minimal state space. The minimal state space for the (set) of RCE will consist of current state variables summarizing the individual state of a representative household and the aggregate state of the aggregate economy. In this

[26] All the proofs of the Lemmas and Theorems in the paper are in the Appendix.

[27] Note that, because of the compactness of the equilibrium, the transversality condition $lim_{t \longrightarrow \infty} \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$ is also satisfied.

[28] Because of the bounds on marginal utility in assumption 1, any equilibrium is compact. Then, the transversality condition $lim_{t \longrightarrow \infty} \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$ can be added to the set of sufficient conditions without loss of generality.

[29] Existence of RCE can be proven without standard Inada conditions. See [64] for a discussion. The presence of Inada conditions, though, facilitates constructing the least RCE by an explicit successive approximation algorithm (e.g., allows us to iterate on our RCE operator from an explicit "lower subsolution" relative to the least RCE).

[30] Notice, to meet Assumption 1(f) and still have an Inada condition as $c^T \to 0$, we can always take period utility to be $U(c) = u(c) + \eta c^T$ for $\eta > 0$ and sufficiently small, and $u(c)$ satisfying the standard Inada conditions

$$\lim_{c \to 0} u'(c) \to \infty; \ \lim_{c \to \infty} u'(c) \to 0$$

economy, in any period of a RE, a household enters the period with an individual level of debt $d \in \mathbf{D} \subset \mathbf{R}$, where $\mathbf{D}$ is compact[31], as well as an endowment of tradeable and non-tradeable denoted by the vector $y = (y^T, y^N)$, where $y \in \mathbf{Y} \times \{\mathbf{y}^{NT}\} \subset \mathbf{R}^2_{++}$. So the *individual state* of the household is characterized by the vector $(d, y) \in \mathbf{D} \times \mathbf{Y}$. Further, a the beginning of any period, a typical household also faces an aggregate economy in an *aggregate state* consisting of per-capita aggregate measures of each of these individual state variables That is, the aggregate state variable is a vector $S = (D, Y) \in \mathbf{D} \times \mathbf{Y} = \mathbf{S}$ is compact, $D \in \mathbf{D}$ is the per-capita level of aggregate debt, Y^T (resp, Y^N) are the per-capita endowment draws for tradeable (resp., non-tradeable) endowments with vector $Y = (y^T, y^N) \in \mathbf{Y} \times \{\mathbf{y}^{NT}\} \subset \mathbf{R}^2_{++}$. Therefore, the state of a household entering any given period in a RCE will be denoted by $s = (d, y, S) \in \mathbf{D} \times \mathbf{Y} \times \mathbf{S}$.

Next, we construct a recursive representation of the aggregate economy on the aggregate state space $S = (D, Y) \in \mathbf{S}$. Anticipating the structure of RCE, the relative price for non-tradeable to tradeable (denoted by $p(C^T)$) in any RCE will be equal to equilibrium marginal rate of substitution between non-tradeable and tradeable: i.e,

$$\frac{U_2(C^T, y^N)}{U_1(C^T, y^N)} = \frac{A_2(C^T)}{A_1(y^N)} = p(C^T) \tag{13}$$

where C^T is some per-capita aggregate level of tradeable consumption, and we impose in any RCE the fact that $c^N = y^N = Y^N$, where Y^T is constant (and we suppress the dependence of p on the assumed constant level of Y^N). Under the supermodularity and concavity conditions in Assumption 1 on preferences relative to the composition $U(A(c))$, we have $p(C^T)$ is increasing in C^T.[32] Using equation (13), we can then generate a recursive representation of sequential prices $p = \{p\{S_t\}\}$ by constructing candidate laws of motion for the per-capita aggregate debt D, which used in conjunction with realizations of endowment $\{Y_t\}$ generate realizations of sequential paths for prices $p = \{p(S_t)\}_{r=}^{\infty}$.

To do this, define a collection of candidate socially feasible per-capita tradeable consumption $C^T : \mathbf{S} \to [0, c^{\max}] \subset \mathbf{R}_+$ denoted by $\mathbf{C}^f(\mathbf{S})$:

$$C^T \in \mathbf{C}^f(\mathbf{S}) = \{C^T(S) | 0 \leq C^T(S) \leq c^{\max},\ C^T \text{ is increasing in } Y, \text{ decreasing in } D, \tag{14}$$

$$\text{and jointly continuous, such that } (1 + \frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R} p(C^T(S))y^N > 0\} \tag{15}$$

where from the previous section, under Assumption 1, $c^{\max}$ is finite. Endow the space $\mathbf{C}^f$ with the standard pointwise partial order $\geq$. Then, the space $(\mathbf{C}^f, \geq)$ is a nonempty partially ordered set. We make a number of remarks about the space $\mathbf{C}^f$ that are critical to our RCE construction ultimately.

First, for $C^T(S) \in \mathbf{C}^f$, the implied aggregate debt mapping $D'(S) = \kappa(Y^T + p(C^T(S))Y^N$ is increasing in S. Second, it is clear we must study the existence of RCE within a *strict subset* of functions $\mathbf{C}^*(S) \subset \mathbf{C}^f(\mathbf{S})$, (where the construction of this subspace $\mathbf{C}^*(S)$ will be discussed in great detail momentarily). There are many reasons for this fact. This is because there exist elements $C^T \in \mathbf{C}^f$ which do not admit *admissible* SCE price sequences $\{p(C^T(S_t)\}$. In particular, for $C^T(S) \in \mathbf{C}^f$, we only impose $0 \leq C^T(S) \leq c^{\max} < \infty$, and as $p(0)$=0 for many parameterizations of preferences, $p(C^T) = 0$ whenever $C^T(S) = 0$ in any state. Therefore, it must be for an RCE price system p satisfy the basic conditions

[31]Under Assumptions 1(a)-(e), and 1(g), we can take $\mathbf{D}$ compact given results on existence of SCE in the previous section.

[32]For example, much of the applied literature uses Armington aggregators for the mapping $A(c)$, so the relative price p is just:

$$p(C) = \frac{1-a}{a} \left(\frac{C^T}{Y^N} \right)^{1/\xi}$$

Under Assumption 1.g, for constants α^T>0 and $\alpha^N > 0$, we could have utility such that the price function is:

$$p(C) = \frac{1-a}{a} \left(\frac{C^T + \alpha^T}{Y^N + \alpha^N} \right)^{1/\xi}$$

such that marginal utility (and hence, prices) are bounded for all $C^T \geq 0$. Note, this special case is is important to keep in mind when relating our multiplicity of RCE results in the sequel to Schmitt-Grohe and Uribe ([74]). Our results do not require Armington aggregators for any of our theoretical results, rather simply preferences which satisfy Assumption 1.

needed for any SCE, any RCE must have $C^{T*}(S) > 0$ for *any* state S, so $p^u \geq p^* \geq p^l > 0$ in all states $S \in \mathbf{S}$.

Along the line of this last remark, the constraint in the definition of the space $\mathbf{C}^f$ requires that C^T be such that $(1 + \frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R}p(C^T(S))y^N > 0$. This is imposed to guarantee a *strict interior point i*n the feasible correspondence for the household optimization problem when the collateral constraint binds. This condition is needed for two reasons: (i) for the existence of a Slater condition for the dual Lagrangian representation of the household's dynamic programming problem, and (b) to guarantee a *strictly positive optimal tradeables policy* (and hence, a strictly positive solution to the "first stage" fixed point problem in Lemma 7). In the second step operator that we use to construct RCE, we will use the monotonicity of our RCE operator to *construct* a lower bound on aggregate tradeables consumption $C_m(S) > 0$ that our operator maps C_m up (i.e, $C_m \leq A^*(C_m)$).

Finally, we emphasize in the existing literature, there is *no* systematic approach to guaranteeing that RCE tradeables consumption is *strictly positive* in all equilibrium states. Indeed, Schmitt-Grohé and Uribe [74] emphasize how difficult guaranteeing this strict positivity condition is even near a steady-state of the deterministic model. In this paper, for RCE, we use an iterative monotone operator approach to the existence question to guarantee that least RCE that are strictly positive (hence, all RCE are strictly positive).

3.1 The Household's Dynamic Programming Problem in a RCE

We can now develop a dynamic programming representation of the household's decision problem. For any element $C^T \in \mathbf{C}^f$, we can identify the implied law of motion for per-capita debt D in a RCE by using equilibrium versions of the household's budget constraints and collateral constraints: i.e., the per-capital debt evolves according to:

$$D' = \Phi(S; C^T) = \inf[R\{C^T(S) - Y + D\}, \kappa\{y^T + p(C^T(S))y^N\}], \ C^T \in \mathbf{C}^f \tag{16}$$

where R is the current interest rate. As D is the only endogenous aggregate state in this economy, in conjunction with the primitives of the stochastic process on the endowment shocks Y, we now have a full characterization of the stochastic transition structure of the aggregate economy in any candidate RCE $C^T(S) \in \mathbf{C}^f$.

A typical household enters any period facing a fixed interest rate $R > 0$ such that $\beta R < 1$,[33] with a current level of individual household debt $d \in \mathbf{D}$, current realizations of endowments $y = (y^T, y^N) \in \mathbf{Y}$, and an aggregate economy in state $S \in \mathbf{S}$ whose continuation aggregate dynamics are parameterized by a single function $C^T \in \mathbf{C}^f$. Then, when entering the period in state $s = (d, y, S)$, the household's feasible correspondence is given by:

$$G(s; C^T) = \{c \in \mathbf{R}^2_+, d' \in \mathbf{D} | \text{ (17a) and (18) hold}\}$$

where

$$c^T + p(C^T(S))c^N \leq y - d + p(C^T(S))y^N + \frac{d'}{R} \tag{17a}$$

and

$$d' \leq \kappa(y^T + p(C^T(S))y^N) \tag{18}$$

For $C^T \in \mathbf{C}^f$, under assumption 1, $G(s; C^T)$ is a continuous correspondence in state $s = (d, y, S) \in \mathbf{D} \times \mathbf{Y} \times \mathbf{S}$. As the aggregate economy is characterized by a law of motion on per-capita debt D' using

[33] In this paper, we only consider the more common case in the literature where $\beta R < 1$. The case of $\beta R = 1$ has been studied sometimes in the literature (e.g., Schmitt-Grohé and Uribe ([74]), but this case is delicate per the question of existence of SCE and/or RE. The problem is compactness relative to *stochastic* dynamics (as opposed existence of deterministic steady states). Similar issues arise when studying the case where $\beta R \geq 1$. In Pierri and Reffett ([64]), we consider both of these cases, and impose sufficient structure of SCE and RE to exist.

(16), a recursive representation of the household's sequential decision problem can be then be constructed as unique value function $V^*(s;C^T)$ solving a Bellman equation for each $C^T(S)\in\mathbf{C}^f$:

$$V^*(s;C^T)) = \max_{x=(c^T,c^N,d')\in G(s;C^T)} U(c^T,c^N)+\beta\int V^*(d',y',Y',\Phi(S;C^T);C^T)\chi(dy') \tag{19}$$

Under Assumption 1(a-e), 1(g), and Assumption 2, for each $C^T\in\mathbf{C}^f$, standard arguments yield the following facts about solutions to equation (19). Noting the strict concavity of the primitive data under Assumption 1(g), the unique optimal policy function associated with the solution to (19) is given by:

$$c^*(s;C^T(S)) = \arg\max_{x=(c^T,c^N,d')\in G(s;C^T)} U(c^T,c^N)+\beta\int V^*(d',y',Y',\Phi(S;C^T);C^T)\chi(dy') \tag{20}$$

where by a standard application of Berge's maximum theorem to the right side of (20) noting the strict concavity of preferences under Assumption 1, the vector of consumption policies $c^*(s;C^T) = (c^{T*}(s;C^T),c^{NT*}(s;C^T))$ are jointly continuous in s, and the value function $V^*(s,S;C^T)$ is continuous in s, strictly concave and decreasing in d for each (y,S), and increasing in y, each (d,S).

We can now formally state the definition of a RCE:

Definition 5 *A minimal state space RCE in this economy is function for per-capita tradeables $C^{T*}\in\mathbf{C}^f$, a household value function $V^*(d,y;D,Y;C^{T*})$ that solves the functional equation in (20) at $C^{T*}\in\mathbf{C}^f$, with optimal solution for consumption $c^*(s,C^{T*}) = (\ c^{T*}(s,C^{T*}),c^{NT*}(s,C^{T*}))$ such that (a) when we have state $s=(d,y,S)\in\mathbf{D}\times\mathbf{Y}\times\mathbf{S}$ $d=D, s=S$, $C^{T^*}(S)\in\mathbf{C}^f$, $C^T(S)>0$ for all $S\in\mathbf{S}$,$c^{T*}(s,C^{T*}(s))$=$C^{T*}(s)>0$, $c^{NT*}(s,C^{T*})=y^N$, and the associated price $p(C^{T*}(s))>0$ and finite, (b) $c^{T*}(s,C^{T*}(s))=0$ else.*

We now make a few remarks about the derivation of the first order theory associated with this dynamic programming problem for representing of the optimal policy function $c^*(s,C^{T*}(s))$. First, relative to the primal dynamic programming problem in ((20)), by appealing to the duality results in Rincon-Zapatero and Santos (([65]), Proposition 3.1 and Theorem 3.1), and noting that for $C^T\in\mathbf{C}^f$ (so we have a strict interior point for consumption in the household's problem), one can check that there exists a an appropriate Slater condition/interior point condition for the existence of sequential dual representation of the household's sequential primal optimization problem, and further there exists a well-defined recursive Lagrangian dual formulation of (20) that sequential dual that can be characterized as follows: for $C^T\in\mathbf{C}^f$, $c\in\mathbf{C}=\{c\in\mathbf{R}^2_+|c^T\in[0,c^{\max}],c^N\in Y^N\}$, $d'\in\mathbf{D}$, we have:

$$v^*(s;C^T) = \inf_{\lambda,\mu\geq 0}\max_{c,d'\in\mathbf{C}\times\mathbf{D}} L(c,d',\lambda,\mu;s,v^*;C^T) \tag{21}$$

where, the recursive dual Lagrangian is given by:

$$\begin{aligned} L(c,d',\lambda,\mu;d,y,S,v^*;C^T) &= U(c^T,c^N)+\beta\int v^*(d',y',\Phi(S;C^T),y;C^T)\chi(dy') \qquad (22)\\ &+\lambda\{y^T+p(C^T(S))y^N+\frac{d'}{R}-c^T-p(C^T(S))c^N-d\}\\ &+\lambda\mu\{\kappa(y^T+p(C^T(S))y^N)-d'\}\end{aligned}$$

where under Assumption 1(a-e), 1(g), and 2, $v^*(s;C^T)=V^*(s,C^T)$, noting the when $C^T\in\mathbf{C}^f$ there is a strict interior point on the feasible correspondence of the the recursive dual problem in (21), the recursive dual is well-defined, finite, strong duality holds with the Lagrangian dual value equal to the Lagrangian primal value, the Lagrangian dual solutions and values equal to those of the (20).

Further, this Lagrangian dual formulation in (22) admits a system of unique system of *stationary* KKT multipliers, $\lambda^*(s;C^T)$ and $\mu^*(s;C^T)$ associated with the infinite horizon sequential dual program that dualizes the household's sequential primal optimization problem in (4) from all initial conditions. Importantly, the associated KKT solutions $\{(\lambda^*(s;C^T),\mu^*(s;C^T))\ ;(c^{T*}(s;C^T),c^{N*}(s;C^T),\ g^*(s;C^T))\}$

in (22) are the unique saddlepoints of (21) dual with value function $V^*(s;C^T) = v^*(s,C^T)$, with the envelope theorem for (20) in d given by:

$$\begin{aligned}\partial_d v^*(s;C^T) &= \lambda^*(s;C^T) \\ &= \partial_d V^*(s;C^T) \\ &= U'(A(c^{T*}(s;C^T), c^{N*}(s;C^T)))A_1(c^{T*}(s;C^T)\end{aligned}$$

where $(c^{T*}(s;C^T), c^{N*}(s;C^T))$ is the vector optimal solutions for consumption goods for the primal dynamic program in (19).

Using these facts, the system of first order conditions (necessary and sufficient) for our problem in (22) can be stated as follows: for stationary KKT multipliers $(\lambda^*(s;C^T), \mu^*(s;C^T))$, the optimal policy functions $c^*(s,C^T) = (c^{T*}(s;C^T), c^{N*}(s;C^T))$, and $d'^*(s;C^T) = g^*(s;C^Y))$ satisfy the following:

$$\lambda^*(s;C^T) = U'(A(c^*(s;C^T))A_1(c^*(s;C^T) \tag{23}$$

$$p(C^T(S)) = \frac{A_1(c^*(s;C^T))}{A_2(c^*(s;C^T))} \tag{24}$$

$$\{\frac{1}{R} - \mu^*(s;C^T)\}\lambda^*(s;C^T) = \beta \int \lambda^*(d', y', \Phi(S;C^T), y'; C^T) \tag{25}$$

$$\{d' - \kappa\{p(C^T(S))y^N + y^T\}\mu^*(s;C^T) = 0, \;\; \mu^*(s;C^T) \geq 0 \tag{26}$$

where the law of motion on individual debt in (25) is given by:

$$d'^*(s;C^T) = \inf\{R\{c^{T*}(s,C^T) - y^T - p(C^T(S))y^N + d\}, \kappa\{y^T + p(C^T(S))y^N\} \tag{27}$$

and the law of motion on per-capita debt D' in (25) is given by $\Phi(S;C^T)$ in equation (16).

3.2 The Structure of Optimal Tradeables Consumption

When developing our RCE operator in a moment, it proves useful to characterize first how the structure of the optimal tradeable consumption policy varies over the two "regimes" (i.e., equilibrium states where the household is collateral constrained versus equilibrium states where household is not collateral constrained). To do this, first note for any household who enters any period in state $s = (d, y, S) \in \mathbf{S}$, after we impose the RCE condition that $c^{*N}(s^e) = y^N = Y^N$, the household's budget constraint in (17a) is:

$$c^{*T} = y^T - d + \frac{d'^*}{R} \tag{28a}$$

If the per-capita tradeable consumption is given by $C^T \in \mathbf{C}^f$, the law of motion on optimal level of debt in equilibrium for the representative household is:

$$d'^*(s;C^T) = \inf\{R\{c^{T*}(s,C^T) - y^T + d\}, \kappa\{y^T + p(C^T(S))y^N\} \tag{29}$$

where $c^{T*}(s;C^T(S))$ is the household's optimal tradeable consumption. Using (23) into (25), we can then define the mapping :

$$Z_p^*(x,s;C^T) = \frac{U_1(x,y^N)}{R} - \beta \int U_1(c^{T*}(d',y',D',Y';\; C^T)\chi(dy') \tag{30}$$

where the evolution of per-capita debt D' in (30) is given by equation (16), and the evolution of individual optimal debt for d' is given by (27). If we let $x^*(s;C^T) > 0$ be the implicit solution of the following:

$$Z_p^*(x^*(s;C^T(S)), s; C^T(S)) = 0$$

which under Assumption 1(a)-(e), 1(g) and Assumption 2 is well-defined as a function continuous in (d, y), the implied optimal debt associated with a tradeable consumption plan $x^*(s; C^{T*})$ would be:

$$d_{x^*}(s; C^T) = R\{x^*(s; C^T) - y^T + d\} \tag{31}$$

If the debt level $d_{x*}(s, C^T)$ in (31) satisfies $d_{x*}(s; C^T) < \kappa\{y^T + p(C^T(S))y^N)\}$, the household is not debt-constrained in state $s = (d, y, S) \in \mathbf{S} \times \mathbf{S}$ for the aggregate tradeable policy $C^T(S) \in \mathbf{C}^f$, as the collateral constraint allows for strictly positive consumption, by the Inada condition, the optimal policy for tradeables consumption will also be positive and given by: [34]

$$c_{uc}^{T*}(s; C^T) = x^*(s; C^T) > 0 \tag{32}$$

Alternatively, if $d_{x*}(s, C^T) \geq \kappa\{y^T + p(C^T(S))y^N\}$, the debt constraint binds (or is saturated) in this state s, and the optimal tradeable consumption

$$c_c^{T*}(s; C^T(S)) = (1 + \frac{\kappa}{R})y^T - d + \frac{\kappa}{R}p(C^T(S))y^N > 0. \tag{33}$$

where strict positive follows from $C^T \in \mathbf{C}^f$ which guarantees in all states (y^T, y^{NT}) the level of debt d must be such that $c_c^{T*}(s; C^T(S)) > 0$. Note here, the Euler equation in (23) binds with $\mu^*(s; C^T) > 0$.

Combining these arguments, the optimal policy for tradeables consumption then has the following form: for any $s = (d, y, S)$:

$$c^{T*}(s; C^T) = \inf\{c_{uc}^{T*}(s; C^T), c_c^{T*}(s; C^T)\} > 0 \tag{34}$$

where the infimum is computed at each (s, C^T) preserves joint continuity over individual states (d, y) for each S to $c^{T*}(s; C^T)$ by a standard application of the Berge's maximum theorem (noting the compactness of the state space). Additionally, one can show $c^{T*}(s; C^T)$ decreasing in d, increasing in y, each $S \in \mathbf{S}$.[35] We can use this structure of the optimal tradeable policy over the "two collateral constraint" regimes to develop our two step monotone approach to constructing RE in the next section.

One final remark: the policy $c^{T*}(s; C^T)$ in (34) is *not* increasing in $C^T(S)$ for each s. That is, the pecuniary complementarities in these models relative to RCE are induced by the equilibrium structure of the price-dependent collateral constraint. Indeed, as we shall show in a moment, the existence of pecuniary complementarities in these models is induced by the collateral constraint *in equilibrium* and cannot be disentangle from the fixed point/equilibrium construction we shall use to construct the set RCE.

3.3 Existence and Comparative Statics of RCE

Motivated by the structure of the optimal tradeable consumption policy in equation (34), we now define our approach to constructing RCE. The challenge of construction a RCE operator models in this model, we need to address the complication of structural change over different equilibrium "regimes" (e.g., collateral constrained states versus unconstrained states). In this since, our equilibrium operator will model RCE tradeable consumption as a "coupled" fixed point of a *two-step* operator that in its definition explicitly keeps track of the two equilibrium regimes that characterize the stochastic equilibrium dynamics of Sudden Stop models. Intuitively, in the first step of this fixed point procedure, we compute the RCE tradeable decision rule *conditional* on holding the "pecuniary externality" *fixed* at the relative price of non-tradeable parameterized by tradeable consumption level $C^T(S)$. This first step mapping will turn out to be a order continuous contraction in an appropriately define partially ordered complete metric space, and will have a unique strictly positive fixed point for each fixed $C^T(S)$. By a fixed point comparative statics result for contractions, this strictly positive fixed point will be continuous in the topology of

[34] If if $d_{x*}(s; C^T) \leq \kappa\{y^T + p(C^T(S))y^N)$, this implies the collateral constraint is either slack or saturated (but not binding) at this state s. This, in turn, implies in equation (25) that the KKT multiplier on the debt constraint is $\mu^*(s; C^T) = 0$, and $Z_*^p(x, s; C^T)$ is the actual FOC for the household after imposing $c^{NT*} = y^N$ in a RE.

[35] In the appendix, see the proof of Lemma 7.

pointwise convergence in $C^T(S)$ (and hence, order continuous in pointwise partial orders).Then, the RCE is computed in the second step noting that this strictly positive fixed point in the first step is order continuous on $C^T(S)$. Therefore, using a order-theoretic fixed point constructions we can compute the "least" and "greatest" RCE collateral constraint consistent with this first step fixed point, which in turn induces the *set* of actual RCE via its coupling with first stage fixed point.

To begin our construction of RCE, motivated by the structure of the household's policy function in this model, assume the structure of the (unknown) RCE tradeable consumption "tomorrow" is given by the following mapping when $d = D$, and $y = Y$:

$$C(c, C^T)(d, y, S) = \inf\{c(d, y), C_c^T(D, Y, C^T(S))\} \tag{35}$$

where in the constrained state, when $C^T \in \mathbf{C}^f$, tradeables consumption is $C_c^T = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(D, Y)Y^N > 0$ and depends only on $C^T(S)$ the per-capital level of tradeable consumption. Then, as with the policy function, the mapping for tomorrow's "consumption" $C(c, C^T)(d, y, S)$ is constructed as the pointwise infimum of two unknown functions: (i) $c(d, y) \in \mathbf{C}^p(S)$, where $\mathbf{C}^p$ is the space of candidate functions for tradeable consumption in states where the household is "conditionally" not collateral constrained, and (ii) $C^T \in \mathbf{C}^f$ which tradeables consumption tomorrow when households are collateral constrained. Per the latter function, notice $C^T(S)$ parameterizes a "guess" at the *equilibrium* collateral constraint (and, hence the per-capital *equilibrium* collateral-constrained consumption $C_c^T(D, Y, C^T(S))$. Given this guess at "tomorrow's" tradeable consumption, we use the household's Euler inequality to compute the implied level of tradeable consumption "today" in equilibrium. We denote this implied mapping "today" by $A(c, C^T))(d, y, S)$ when $d = D$ and $y = Y$.

We will solve the resulting functional equation in two steps. To formalize our construction, we begin by defining the first step domain of our two-step RCE operator.[36] For the first step of our construction, we shall fix the "second step" function at $C^T \in \mathbf{C}^f$, and take the domain of our the operator $A(c; C^T)(d, y, S)$ the set of functions $c = c(d, y) \in \mathbf{C}^p(\mathbf{D}^e \times \mathbf{Y}^e)$, where:

$$\begin{gathered}\mathbf{C}^p(\mathbf{D}^e \times \mathbf{Y}^e) = \{c(d, y) | 0 \leq c(d, y) \leq y^T - d + (d^{Max}/R), \\ c(d, y) = \tilde{c}(d, y, d, y), \ \tilde{c}(d, y; d, y) \text{ decreasing in } d, \\ \text{increasing in } y, \\ \text{s.t. } (*) \text{ for a fixed } y, \ -d' = R(y^T - d - c(d, y, d, y)) \text{ decreasing in } d, \text{ increasing in } y\}\end{gathered} \tag{36}$$

where d^{Max} is the maximal level of debt as shown in section 2, and we use the notation $s^e = (d, y, d, y) \in \mathbf{D}^e \times \mathbf{Y}^e$ to make clear the domain of any RE function is defined over a *diagonal* of individual state variable $s = (d, y, S)$:

$$\mathbf{D}^e \times \mathbf{Y}^e = \{(d, y) | \ (d, y) = (d, y, D, Y), \ d = D, \ y = Y\} = \mathbf{S}$$

where by construction $s^e \in \mathbf{D}^e \times \mathbf{Y}^e = \mathbf{S}^e$ is an *equilibrium* state of the household. We endow $\mathbf{C}^p(\mathbf{S}^e)$ with its relative pointwise partial order.

Lemma 6 *The space $\mathbf{C}^p(\mathbf{S}^e)$ is: (a) an equicontinuous collection of continuous functions (b) a nonempty complete lattice in its relative pointwise partial order.*

We remark that for any function $c(d, y) \in \mathbf{C}^p$, the implied policy function for debt $d'(d, y)$ is *decreasing* in y, and *increasing* in d, so the elements of the space $\mathbf{C}^p(\mathbf{D}^e \times \mathbf{Y}^e)$ are consistent with the fact that when the household is *not* debt constrained, so $d'(d, y)$ works as a consumption smoothing device for household relative to the (tradeables) endowment shocks.

To define our two-step RCE operator then, we first rewrite Euler inequality Z_p^* in (30) for the household in a RCE as follows: for any $c(d, y) \in \mathbf{C}^p$, when $c(d, y) > 0, \ C^T \in \mathbf{C}^f$, using $C(d, y; c, C^T(d, y))$ is defined

[36]Note, the definition of the second-step domain then will "couple" with the definition of the first step operator and its domain. For the moment, we simply take the second-step domain to have $C^T \in \mathbf{C}^f$ where $\mathbf{C}^f$ was defined earlier.

in (35), imposing equilibrium between individual and aggregate states $d' = D'$ and $y' = Y$", we can define the mapping Z^*_{uc} which computes the equilibrium tradeables consumption solution if in the equilibrium state space by $s^e = (d, y, d, y) \in \mathbf{S}^e$ the household is not collateral constrained:

$$Z^*_{uc}(x, s^e; c, C^T), = \frac{U_1(x, y^N)}{R} - \beta \int U_1(c(R(x - y^T + d), y', C^T_c(R(x - y^T + d), y')\chi(dy') \tag{37}$$

where we obtain Z^*_{uc} from Z^*_p in (30) as follows: (a) we have assumed s^e is *not* collateral constrained (which we will discuss how to verify when this assumption is correct in a moment), and (b) set $x = C^T(S)$ pointwise in equilibrium on Z^*_p for the updating of the aggregate debt state in the it's aggregate law of motion. Relative to (b), this substitution is *critical* as delivers precisely the appropriate *equilibrium* single crossing condition between the household's tradeable consumption and the aggregate level of tradeable consumption in any RCE that will allow use to obtain a monotone operator for computing RCE.

Next, to define then our two-step RCE operator, for any $c(d, y) \in \mathbf{C}^p, c > 0$, and $C^T \in \mathbf{C}^f$, as $Z^*_{uc}(x, s^e; c, C^T)$ is strictly decreasing in x, increasing in (c, C^T, y), and decreasing in d, and we can compute function $x^*_{uc}(s^e; c, C^T) > 0$ when $c(d, y) > 0$ implicitly in this expression:

$$Z^*_{uc}(x^*_{uc}(s^e; c, C^T), s^e; c, C^T) = 0 \tag{38}$$

which is well-defined as a function as Z^*_{uc} is strictly decreasing under Assumption 1, is continuous in its first two arguments, all the parameters, and pointwise continuous (topology of pointwise convergence) in (c, C^T) for each s^e,[37] and under Assumption 1 plus Assumption 2, the root $x^*_{uc}(s^e; c, C^T) > 0$. By a standard comparative statics argument, under Assumptions 1 and 2, this root $x^*_{uc}(s^e; c, C^T)$ is increasing in (c, C^T, y) and decreasing in d.

Then following the construction in the previous section for the optimal tradeable consumption policy, the implied debt level associated with the tradeable consumption $x^*_{uc}(s^e; c, C^{T*})$ will be:

$$d_{x^*_{uc}}(s^e; c, C^T) = R\{x^*_{uc}(s^e; c, C^T) - y^T + d\} \tag{39}$$

If the debt level $d_{x^*_{uc}}(s^e, c, C^T)$ in (39) satisfies $d_{x^*_{uc}}(s^e; c, C^T) \leq \kappa\{Y^T + p(C^T(S))Y^N)$, the household is not debt-constrained state $s^e = (d, y, d, y) \in \mathbf{S} \times \mathbf{S}$ for (c, C^T),[38] and we have:

$$A_{uc}(c; C^T)(s^e) = x^*_{uc}(s^e; c, C^T) > 0 \tag{40}$$

where strict positivity of $A_u c(c; C^T)(s^e)$ follows from the fact that for $C^T \in \mathbf{C}^f$ strictly positive condition is possible in all states, and hence in the unconstrained states, the Inada condition implies $x_u c^*(s^e; c, C^T) > 0$.

Alternatively, if $d_{xu}c^*(s, C^T) > \kappa\{y^T + p(C^T(S))y^N$, the debt constraint binds in state s, and the optimal tradeable consumption is

$$A_c(C^T)(S) = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N > 0. \tag{41}$$

Then our operator for $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$ will be defined as:

$$\begin{aligned} A(c, C^T)(s^e) &= \inf\{A_{uc}(c, C^T)(s^e), A_c(C^T)(S)\} \text{ when } c > 0 \\ &= 0 \text{ else} \end{aligned} \tag{42}$$

We now are ready to construct the set of RCE in two steps. In Lemma 7, we show for each *fixed* $C^T \in \mathbf{C}^f$, there is a *unique strictly positive* fixed point $c^*(C^T)(s^e)$ for tradeables consumption of partial mapping $A(c; C^T, \beta, \kappa, R)(s^e) \in \mathbf{C}^p$ (where in the Lemma, we now make explicit how the operator varies in the deep parameters (β, κ, R)) eventually RE comparative statics will be of interest in our main theorem).

[37]For the situation that $c = c(d, y) = 0$ in any state, take $x^*_{uc}(s^e; c, C^T)$=0. We should also remark, when computing the of implied debt for $x^*_{uc}(s^e; c, C^T)$ when $c = c(d, y) > 0$, if the collateral constraint is not binding, $x^*_{uc}(s^e; c, C^T) > 0$.

[38]Notice, when the debt $_{x^*_{uc}}(s^e; c, C^T) = \kappa\{Y^T + p(C^T(S))Y^N)$, the collateral constraint is saturated, but not binding. In this case, the implied KKT multiplier on the collateral constraint would be 0.

Lemma 7 *Under Assumptions 1(a-e), 1(g), and 2, for each $C^T(S) \in \mathbf{C}^f$, and $S \in \mathbf{S}$, for the operator $A(c; C^T, \beta, \kappa, R)(s^e)$, (a) there exists a unique strictly positive fixed point $c^*(d, y, C^T(S), \beta, \kappa, R)(s^e)$ in $\mathbf{C}^p(S)$; (b) this fixed point can be computed by the decreasing chain of successive approximations* $\inf A^n(c_{\max}; C^T(S), \beta, \kappa, R)(d, y) \searrow c^*(C^T(S), \beta, \kappa, R)(s^e)$; *(c) $c^*(C^T, \beta, \kappa, R)(s^e)$ is a monotone operator on $\mathbf{C}^f(\mathbf{D}^e \times \mathbf{Y}^e)$, (d) $c^*(C^T, \beta, \kappa, R)(s^e)$ is decreasing in (β, R), and increasing in κ.*

Next, construct a new operator using the fixed point comparative statics of the first-step (unique) strictly positive fixed point $c^*(C^T, \beta, \kappa, R)(d, y)$. The domain for second step mapping will be denoted $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, and defined as follows:

$$C^T \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e) = \{C^T(d,y) | C^T \in \mathbf{C}^f,\ C^T \in [C_m(S), c_{\max}] \tag{43}$$

$$C(d, y, d, y) = \inf\{c^*(d, y, C^T(D, Y)),\ C^T(D, Y)\},\ d = D, y = Y$$

$$c(d, y; C^T(D, Y)) \in \mathbf{C}^{p*}(d, y) \text{ for fixed } (D, Y),\ C^T(D, Y) \in \mathbf{C}^f \text{ st } D'(D, Y) = \kappa(Y^T + p(C(D, Y))Y^N\ \} \tag{44}$$

$$\text{is increasing in } Y,\ \text{decreasing in } D\} \tag{45}$$

where $C_m(S)$ is a *strictly positive* function (hence, any fixed point in $\mathbf{C}^*$ has strictly positive tradeables consumption *in all states). O*bserve the elements of the space $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e) \subset \mathbf{C}^f(\mathbf{D}^e \times \mathbf{Y}^e)$ have the exact same structure as those in $\mathbf{C}^p$ over *individual states*, but drops the condition (*) in (36) governing RE debt dynamics when collateral constrained binds. This reflects the fact that when the collateral constraint binds in equilibrium, the properties of the implied RE debt dynamics *reverse in order* to those implied by $c^*(d, y, C^T(S)) \in \mathbf{C}^p$ (the unconstrained regime for fixed C^T).

We now state an important property of $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$

Lemma 8 $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ *is a nonempty complete lattice.*

To define the second step operator we use the unique positive fixed point of the first step operator, and define the following nonlinear operator on the space $\mathbf{C}^*$: when $(d, y) = (D, Y) = S$, $C^T \in \mathbf{C}^*$,

$$A^*(C^T; \beta, \kappa, R)(s^e)) = \inf\{c^*(C^T; \beta, \kappa, R)(s^e),\ A_c(C^T; \kappa, R)(s^e)\} \tag{46}$$

where we now make the dependence of the operator A^* on the deep parameters (β, κ, R) explicit. We now have the following result.[39]

Theorem 9 *Under Assumption 1(a-e), 1(g), and 2, there exist (i) a nonempty complete lattice $\Psi^*(\beta, \kappa, R)$ of RCE for tradeables consumption in C^* (i.e., a nonempty complete lattice of strictly positive tradeables consumptions in C^*) , with the least and greatest elements of $\Psi^*(\beta, \kappa, R)$. Further, the least and greatest RCE can be computed by successive approximation as follows:*

$$\begin{aligned} 0 &< \inf_n A^{*n}(C_m; \beta, \kappa, R)(s^e) \to C_l^*(\beta, \kappa, R)(s^e)) = \wedge\Psi^*(\beta, \kappa, R)(s^e) \\ &\leq \vee\Psi^*(\beta, \kappa, R)(s^e) = C_g^*(\beta, \kappa, R)(s^e) = \sup_n A^{*n}(c_{\max}; \beta, \kappa, R)(s^e) < c_{\max} \end{aligned}$$

where both least RCE $\wedge\Psi^*(\beta, \kappa, R)(s^e) = C_l^*(\beta, \kappa, R)(s^e))$ and greatest RCE $\vee\Psi^*(\beta, \kappa, R)(s^e) = C_g^*(\beta, \kappa, R)(s^e))$ are each increasing in κ, and decreasing in (β, R), and the lower subsolution C_m is the least element of $\mathbf{C}^*$ (i.e., $C_m = \wedge\mathbf{C}^*$.)

[39] Using the recent results generalizing the Tarski-Kantorovich theorem to *any* initial iterate in the papers of Olszewski [58] and Balbus, Olszewski, Reffett, and Wozny ([8]), in our setting we can provide *tight fixed point bounds* on iterations of our fixed point operator $A^*(C)$ from *any initial* $C_0 \in \mathbf{C}^*$. We discuss related matters when we provide iterative monotone RCE comparative statics in Theorem 10 later in this section.

We make a critical remark about the proof of Theorem 9. Using the monotonicity of our operator $A^*(C;\beta,\kappa,R)(s^e)$, we are able to *construct* by successive approximations of our operator $A^*(C;\beta,\kappa,R)(s^e)$ from the function $C^T=0$ and any initial state s^e such that $\wedge y^T-d\geq 0$ the least element C_m in the definition of the space $\mathbf{C}^*$. This fact is of critical importance in applied work as we provide an explicit (and intuitive) computational method for computing the least element of the space $\mathbf{C}^*$. Importantly, it turns out that given the order continuity of our operator $A^*(C;\beta,\kappa,R)(s^e)$, we then can show that the element $C_m=\wedge\Psi^*(\beta,\kappa,R)(s^e)$ (the least RCE). Hence, our approach in the paper provides a *systematic* approach guaranteeing that *all* RCE are associated with *strictly positive tradeables consumptions.*

Theorem 9 provides sharp RCE comparative statics on the "low" and "high" borrowing equilibria for this class of economies. These equilibrium comparative statics results are obtained by an application of standard Tarski-Kantorovich principles applied to the operator $A^*(C^T;\beta,\kappa,R)$. Our RCE results on least and greatest RE formalize the multiplicity result in Schmitt-Grohé and Uribe ([73], [74]). What is clear from our approach is that relative to the set of RCE is that the issues raised in Schmitt-Grohé and Uribe ([73], [74]) are *global,* and not related per sa to the deterministic steady-state. That is, from *any state* where collateral constraints bind (not just near steady state), in the stochastic RE (not just the deterministic version of the model), multiplicities of RCE are possible (and, in particular, "low" and "high" borrowing equilibria are possible).

We can see for the case of RCE how their argument works in a global sense. In particular, in any state equilibrium state s^e where the collateral constraint binds, we have our operator $A^*(C^T)(s^e)$ given by

$$A^*(C^T)(s^e)=A_c(C^T)(s)=(y^T+y^{NT}p(C^T(S)))(1+\frac{\kappa}{R})-d$$

Consider the following mapping when $d=D$, $y=Y$:

$$Z_c^*(x^*(d,y,S,C^T),\ C^T(S);d,y,S)=x^*(d,y,d,y,C^T)-(y^T+y^{NT}p(C^T(S)))(1+\frac{\kappa}{R})-d=0$$

where our operator is defined as $A_c(C^T)(s^e)=x^*(d,y,d,y;C^T)$ when the collateral constraint binds in a state s^e. Further, evaluating Z_c^* at $A_c(C^T)(s^e)=x^*(d,y,d,y)=C^{T*}(d,y)$ when $d=D$ and $y=Y$, and now noting the dependence of the mapping on the ratio parameters κ/R, we see that unless $p(\cdot)$ is such the for all parameters κ, R, and y^N the mapping

$$Z_c(x,x,y,d;\kappa/R)=x-(y^T+y^{NT}p(x))(1+\frac{\kappa}{R})-d=0 \tag{47}$$

has *unique* roots $x^*(d,y,d,y;\kappa/R)\geq 0$, there will be *multiple RCE* at this state s^e.[40] Also, notice in the case of multiple roots $x^*(d,y,d,y;\kappa/R)$ to equation 47, if we define the correspondence

$$X^*(d,y,d,y;\kappa/R)=\{x^*(d,y,d,y\geq 0|Z_c(x,x;y,d;\kappa/R)=0\}$$

at equilibrium state s^e where the collateral constrain binds, this correspondence $X^*(d,y,d,y;\kappa/R)$ can easily be show to be well-defined (e.g., by an application of the intermediate value theorem) and have a least and greatest element (as the correspondence $X^*(d,y,d,y;\kappa/R)\subset\mathbf{R}_+$ is chain-valued, nonempty and compact-valued under assumption 1 and 2, hence has a least and greatest element). This also implies in equilibrium states s^e where the collateral constraint binds, as collateral constraints are price-dependent and $p(C)$ is increasing in C, the equilibrium collateral constraints will be *ordered* relative to least and greatest RE tradeable consumption levels. We show then in our main theorem above that as Schmitt-Grohé and Uribe ([74]) suggest, for RCE, we will have (globally) "low borrowing" (associated with "least" RE tradeable consumption) and "high borrowing" (associated with "greatest" RE tradeable consumption) in any equilibrium state s^e, and these least and greatest RE will be distinct in states where the equilibrium collateral constraint binds and the correspondence $X^*(d,y,d,y;\kappa/R)$ is not a single-valued.

[40] Schmitt-Grohé and Uribe ([74]) give a *local* sufficient condition near the deterministic steady-state for this to be case for the case that the utility aggregator $A(c^T,c^N)$ is an Armington aggregator and near a steady-state. But clearly, their idea about the source of multiplicity applies in *any* equilibrium state s^e. That is, generally $Z_{cc}(x,x;y,\kappa/R,d)$ is not either strictly increasing or decreasing in x at each s^e under Assumption 1 (hence, roots are unique)

We conclude our discussion of existence of RCE by showing that we can construct RCE comparative statics for *any* RCE $C^*(\beta,\kappa,R)(s^e) \in \Psi^*(\beta,\kappa,R)(s^e)$. We do this using the results in a recent paper by Balbus et al ([9]) who propose a generalized iterative procedure for order continuous operators equation in complete lattices that provide comparative fixed point bounds for any fixed point in the deep parameters of the operator. The results in that paper apply in our context (e.g., see ([9], Proposition 2).

More specifically, consider the following lower iterative process from any initial fixed point $C^*(\beta,\kappa,R) \in \Psi^*(\beta,\kappa,R) \subset \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, where $C_0 = C_\wedge^{0,\gamma} = C_\vee^{0.\gamma} = C^*(\beta,\kappa,R)$ for all $\gamma \in \mathbb{N}$. Then, compute the following sequences of iterations:

$$C_\wedge^{k+1,\gamma} = A^*(\inf\{C_\wedge^{k,\gamma},...,C_\wedge^{k-\gamma,\gamma}\};\beta',\kappa',R')$$

where for $l>k$, just assume $C_\wedge^{k-l,\gamma} = C_\wedge^{0,\gamma}$. Then, for any γ, compute the order limits:

$$\liminf_k C_\wedge^{k,\gamma} = C_\wedge^*(\beta',\kappa',R') \in \Psi^*(\beta',\kappa',R')$$

where the limit exists in $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ as $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ is complete lattice, and $C_\wedge^*(\beta',\kappa',R') \in \Psi^*(\beta',\kappa',R')$ is a fixed point $A^*(C^T;\beta',\kappa',R')$ as A^* is order continuous. Then we have the following iterative monotone comparative statics result for *any* initial RCE $C^*(\beta,\kappa,R) \in \Psi^*(\beta,\kappa,R)$ at the parameters (β,κ,R):

Theorem 10 *For any $C^*(\beta,\kappa,R) \in \Psi^*(\beta,\kappa,R) \subset \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, and $(R',\beta') > (R,\beta)$, and $\kappa' > \kappa$, $C^*(\beta,\kappa,K) \leq C_\wedge^*(\beta',\kappa',R') \in \Psi^*(\beta',\kappa',R')$.*

3.4 Uniqueness of RCE

We now consider the question of sufficient conditions for uniqueness of RCE in this model. For this, we reconsider Lemma 7 and Theorem 9 under the following additional assumption:

Assumption 3: Assume the consumption aggregator $A(c)$ is such that for the associated $p(x) = \frac{A_1(x,y^N)}{A_2(x,y^N)}$, the mapping $Z_c(x,x;y,d,\kappa/R) = 0$ in equation 47 has a unique root $x^*(d,y,d,y;\kappa/R)$.

Under Assumptions 1-3, we can now show (as a corollary of Lemma 7) we can sharpen our existence theorem on RCE (Theorem 9) to show RCE is unique. Before we state our result, first solve the collateral constraint for the level of tradeables consumption imposed in any RCE by a binding collateral constraint in any equilibrium state $s^e = (d,y,d,y)$ by using equation 47: that is, compute pointwise $c_c^{T*}(d,y) = x^*(d,y,d,y)$ in

$$\begin{gathered} Z_{cc}(x^*(d,y,d,y),x^*(d,y,d,y),y,d) = \\ x^*(d,y,d,y) - (y^T + y^{NT}p(x^*(d,y,d,y)))(1+\frac{\kappa}{R}) - d = 0 \end{gathered} \tag{48}$$

which is unique by Assumption 3. Then, we have the following theorem

Theorem 11 *Under Assumption 1(a-e), 1(g), 2, and 3, the unique RCE is $c^*(d,y) = \inf\{c^{T*}(d,y), c_c^{T*}(d,y)\}$ where for any $c \in \mathbf{C}^p, c > 0$, with*

$$A^{*n}(c)(d,y) = c^{T*}(d,y)$$

.

A few remarks on our uniqueness result. First, in the sequel of the paper, section 5, we will discuss examples of aggregators $A(c)$ such that Assumption 3 is satisfied (e.g., $A(c)$ is quasi-linear or log). We do want to emphasize that in the existing literature when quantitative versions of this model are studied, $A(c)$ is Armington/CES aggregator. In this case, for typical parameterizations of this mapping, Assumption

3 will *not* be satisfied, and multiple RCE will exist (e.g., see the excellent discussion in Schmitt-Grohé and Uribe ([74])).

Second, when using our two-step construction in theorem 4, our iterations on our fixed point operator (which was defined at the second-step of the construction) could start from *any* C^T in the space $\mathbf{C}^*$ (including $C^T = 0$). The reason is in the first step construction, the unique fixed point of the mapping was strictly positive in all states (imply the second step fixed point operator had no trivial fixed points).

In our uniqueness argument for Theorem 11, things are different. Because of Assumption 3, we know what the maximal consumption is in any collateral constrained RCE (i.e., $c_c^{T*}(d, y)$). So the second step of the RCE construction is not needed. In this case, everything can be done with the first step operator which just constructs the unique $c^{T*}(d, y)$ which is the RCE consumption in the "unconstrained" states. Then the RCE consumption is the infimum pointwise by state over the two consumption functions.

4 Generalized Markov Equilibrium (GME)

The previous section proves the existence of a RCE representation of SCE that is time-invariant and defined on a *minimal state space*. In this section we will explore the difference between GME and RCE. The GME approach will have important implications from a numerical perspective. GME representations of SCE differ substantially from those in section 3, and they are deeply connected to recursive representations in Duffie, et. al. ([31]) or Feng, et. al. ([33]).

While the results in section 3 can be used to numerically approximate the SCE and to perform accurate numerical comparative statics exercises, we have been silent about simulations in these class of models. As discussed in Santos and Peralta- Alva ([67]), the starting point of any simulation experiment for a stochastic dynamic equilibrium model is an appropriate *stochastic steady state* notion. The most frequent stochastic steady notion used in the literature is an *invariant measure* (IM) for *some* recursive representation of the sequential equilibrium. Heuristically, an IM gives a sense of *probabilistic time invariance*. That is, if $\{x_t\}$ is a sequence of random variables generated from some Markov process and x_t is distributed according to an IM μ, then x_τ will be distributed according to μ for $\tau > t$.

This section introduces the notion of GME, which has an expanded state space when compare to the results in section 3. This equilibrium is a slightly modified version of the one in Feng, et. al. ([33]). We carefully select the additional state variable with respect to the RCE to guarantee the existence of an appropriate stochastic steady state. Moreover, a GME is defined using the set of equations characterizing the SCE. Thus, the additional state variable and the direct connection with the SCE brings more memory into the model at the cost of allowing additional sources of equilibrium multiplicity. This paper proves that it is possible to refine the equilibrium set in a GME by picking a selection which insures the existence of an IM. In this sense, we are imposing long-run restrictions in order to refine the equilibrium set.

Provided a stationary recursive representation, the existence of an IM implies that simulations obtained from the SCE can be approximated by a time invariant and finite set of functions, abstracting from numerical errors. [41] This is possible by means of a *law of large numbers*, which in turn requires the IM to be *ergodic*. From a practical perspective, ergodicity insures roughly speaking that "averages converges". That is, the existence of a stationary recursive representation, as the one derived in section 3, is not enough to insure the desired convergence. For this purpose, we must use a well defined (i.e., ergodic) law of large numbers. The ergodicity of the IM guarantees that the Cesaro average of any simulation starting from a "nice" initial condition will converge to an *expected value* computed using the stochastic steady state distribution. This last fact allows to connect the model with observed (time independent) stylized facts.

As discussed in Pierri ([61]), existence of an IM and its ergodicity in this type of models are related to the cardinality of the set of exogenous shocks. We prove the existence of an ergodic IM for economies with a *finite set of shocks* under milder assumptions than those studied in Pierri ([61]). This is possible because of the monotonicity properties of the (minimal state space) RCE in endowment shocks combined with an occasionally binding (collateral) constraint. Under this setting it is possible to show that the

[41] As pointed out in Santos and Peralta Alva ([67]), truncation and interpolation errors could accumulate over time if they are not "controlled".

model described in section 2 has an "irreducible atom", which is used in the stochastic process literature to prove the existence of an IM. The ergodicity of this measure follows from its uniqueness. While these results are enough to insure convergence in the sense implied by a law of large numbers, finding the appropriate set of initial conditions maybe problematic as the process may have divergent paths. Fortunately, we can characterize the set of appropriate initial conditions and at the same time prove the existence of an ergodic IM.

4.1 A convenient recursive representation

In this subsection we derive the set of Generalized Markov Equilibria. As this equilibrium have a bigger state space when compared with the recursive representation presented in section 3, it is more flexible (see Kubler and Schmedders ([40] for a discussion). We are interested in preserving the structural properties (i.e., differentiability of the value function, etc.) proved before as they will be useful to derive the results in this section. One of the purposes of writing the minimal state space equilibrium is to refine all possible GME in order to build a selection which replicates the observed behavior. It turns out that, if we restrict attention to a finite set of shocks, it is easy to characterize a "regeneration point" for the global stochastic dynamics in the model using the GME. This is a first step in order to find a recurrent structure that is robust to the presence of multiple equilibria, which typically generate discontinuous selections. The properties of the endogenous variables derived in the RCE for the unconstrained case (i.e. when the collateral restriction does not hold with equality) will be useful to construct trajectories with positive probability which can be used to prove ergodicity.

As can be seen in section 2, any SCE can be characterized using a set of primal first order conditions (as inequalities) which do not depend on Lagrangian multipliers. The usefulness of this representation will be clear in this subsection. Moreover, the existence of well behaved envelopes for the value function in the RCE implied that a SCE can be characterized recursively by the following equations:

$$[-A1(c^T)p + A_2(c^N)] = 0 \tag{49}$$

$$[\kappa\{y^T + py^N\} - d_+][U'\{A_1(y^T + d_+R^{-1} - d)\} - E(m_+)] = 0 \tag{50}$$

$$d_+ \leq \kappa\{y^T + py^N\} \tag{51}$$

where $m = \frac{\partial\ V}{\partial\ d}$ is the envelope of the value function in the household's problem of the minimal state space representation of SCE, with $U' \equiv\ U'(A(y^T + R^{-1}d_+ - d; y^N))$. Given the compactness of the equilibrium set, the results in Feng, et. al. imply that equations (49) and (50) can be used to derive a correspondence, Φ, the so-called *equilibrium correspondence*, which contains the entire set of GME representations of SCE, where $\Phi : Z \times\ Y \longmapsto\ Z$ with $z = [d \quad y^T \quad y^N \quad c^T \quad c^N \quad p \quad m]$ with $z \in Z$, $y^T \in Y$ and Z compact.

Notice, we are restricting m to be an envelope of the value function from some RCE. This restriction is required to show ergodicity as the proof involves a path for d_+ with some qualitative properties which follow directly from the optimization problem in the RCE. In this sense, the choice of an additional state variable is critical. Moreover, the results in section 3 insure that equation (50) under this restriction can be used to characterized any SCE. Thus, when the (collateral) constraint hits, we know that $U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)\} \geq\ E(m_+)$ and the equilibrium for any given period can be computed using the following set of equations:

$$p = \frac{A_2(y^N)}{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)} \tag{52}$$

$$c^T = y^T + R^{-1}\kappa\{y^T + py^N\} - d \tag{53}$$

$$c^N = y^N \tag{54}$$

$$m_+ = U' A_1(c_+^T) \tag{55}$$

$$U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)\} \geq\ E(m_+) \tag{56}$$

Given (y^T, d), the remaining variables in z can be computed using (52) to (55) as long as $U'\{A_1(c^T)\} \geq E(m_+)$. As the recursive equilibrium notion in Feng, et. al. is computed "backwards" (i.e. given "z_+" we obtain z), the Euler equation imposes a looser restriction to the system when compared to the non-binding case. This fact turns out to be very useful to prove the ergodicity of the process in the finite state space case. We are now in position to formally define a GME.

Definition 12 *Generalized Markov Equilibrium (GME) Let Z be the compact set which contains any state z_s that solves* (49)*,* (50) *and* (51) *backwards with $s = 0, 1, \ldots$. The equilibrium correspondence Φ mapping $Z \times\ Y \to Z$ can be defined as follows: let $L = 0$ be the system formed by equations* (49)*,* (50) *and* (51)*. Note that any vector $(z_s, z_{s+1}(y_{LB}^T), \ldots, z_{s+1}(y_{UB}^T))$, where $y_{s+1}^T \in \left\{y_{LB}^T, \ldots, y_{UB}^T\right\}$, satisfies $L(z_s, z_{s+1}(y_{LB}^T), \ldots, z_{s+1}(y_{UB}^T)) = 0$. Then, $z_{s+1}(y_{s+1}^T) = \varphi\left(z_s, y_{s+1}^T\right)$, where $\varphi \in \Phi$ is a selection of the equilibrium correspondence. We say that φ is a GME. Any φ that is independent of time is a stationary GME. Let (Z, P_φ) defines a stationary Markov process with kernel P_φ. If (Z, P_φ) has an ergodic invariant measure, we say that φ is ergodic.*

Given the compactness of Z, as there are a finite number of shocks, the measurability requirements for P_φ follows from Feng, et. al. ([33]).

4.2 Stationarity and Ergodicity under a finite set of shocks

We now derive the set of stochastic steady state for the model. Formally, we show that Φ has an ergodic selection: if we restrict the number of possible distinct values that y^T can take to be finite, we can prove the existence of an ergodic probability measure associated with a selection φ of Φ. In this framework, equation (56) can be used to construct a *point* d_* that generates a set which the process hits with positive probability starting from any initial condition. This point will be called *atom* and solves the system of equations given by (52) to (56) for wealth d_*, with the Euler equation holding with equality and for the lowest possible level of y^T. Moreover, the process will hit the atom in finite time. Thus, it creates an orbit which endows the dynamical system with a recurrent structure, which in turn implies that there will be a unique (and thus ergodic) invariant measure for each atom.

Once we find d_*, we will construct a stable state space. That is, any meaningful (i.e. with positive measure) subset of this state space will be hit by the process in finite time. This property, called irreducibility, will insure the uniqueness and ergodicity of the process. The existence of 2 regimes, one defined for equations (49) and (50) when the collateral constraint does not bind and the other given by equations (52) to (56), together with the possible multiple solutions to equation (52) suggests the presence of multiple possibly discontinuous Markov equilibria. If we allow for discontinuous selections of the equilibrium correspondence in the GME, we can construct a transition function that "jumps" to the atom every time the collateral constraint is hit, generating a "crises". Thus, the presence of multiple equilibria, which is in part a consequence of the long-term memory inherited from the SCE, and its implications for the smoothness of the selections $\varphi \in \Phi$ increases the predictive power of the model in the sense that it allows a better match of long run empirical regularities due to the ergodicity of the equilibrium. Thus, it is critical to understand the "anatomy" of the equilibrium set, something that is done in section 5.1 and 5.2. As suggested by Stokey, Lucas and Prescott ([78]) [42], the existence of such

[42] See for instance exercise 11.4, Ch. 11.

a point is enough to derive a stationary Markov process. The results in Meyn and Tweedie ([53]) give us the tools to prove all the intermediate steps required to go from the existence of an stationary selection of the equilibrium correspondence to its ergodicity.[43] Our identification of an "atom" in this paper is related to the presence of occasionally binding constraints in non-optimal general equilibrium economies, and our application of these tools is novel in the literature. Thus, we will prove the results step by step as it is immediate to extend the methodology used in this paper to another related frameworks with equilibrium collateral constraints.

In any RCE, the model is characterized using the Markov kernel and, thus, in 1 "step". This is a consequence of the "short memory" approach in this type of equilibrium. The GME, as it is computed directly from the sequential equilibria, allows us to bring "memory" into the picture. That is, it is possible to construct a *finite time path from the sequential equilibria*, which guarantees the existence of an ergodic Markov process.

Let us start by formally defining an "accessible atom", which can be thought as a point that is non-negligible from a probabilistic perspective and gets "hit" frequently. Let $\varphi \sim \Phi$ be a selection of the equilibrium correspondence defined in the previous subsection. The compactness of $Y \times Z$ and Z guarantees the measurability of φ. [44] Further, $P_\varphi(z, A) \equiv \{p(y^T \in Y : \varphi(z, y^T) \in A)\}$ defines a Markov operator (i.e. $P_\varphi(., A)$ is measurable and $P_\varphi(z, .)$ is a probability measure) and (Z, P_φ) a Markov process where y^T is assumed to be iid with probability $p(y^T)$. [45] Let $P^n_\varphi(z, A)$ be the probability that the Markov chain goes from z to any point in A in n steps with A being measurable, let ψ be some measure, and $B(Z)$ be the Borel sigma algebra generated by Z. Then the set $A \in B(Z)$ is *non-negligible* if $\psi(A) > 0$. A chain is called *irreducible* if, starting from any initial condition, the chain hits all non-negligible sets with positive probability in finite time (i.e., $\psi(A) > 0 \longrightarrow P^n_\varphi(z, A) > 0$). [46] Intuitively, irreducibility is a notion of connectedness for the Markov process as it implies non-negligible sets are visited with positive probability in finite time.

We are now in position to define an atom and state an important intermediate result.

Definition 13 *Accessible Atom. A set $\alpha \in B(Z)$ is an atom for (Z, P_φ) if there exists a probability measure ν such that $P_\varphi(z, A) = \nu(A)$ with $z \in \alpha$ for all $A \in B(Z)$. The atom is accessible if $\psi(\alpha) > 0$.*

Intuitively an atom is a set containing points in which the chain behave like an iid process. Any singleton $\{\alpha\}$ is an atom. Note that there is a trade off: if the atom is a singleton, the iid requirement is trivial but, taking into account that the state space is uncountable, the accessibility clause becomes an issue as it is not clear how to choose ψ. For instance, the typical measure, the Lebesgue measure, it is not useful as any singleton has zero measure in it. The same happens with irreducibility: when the state space is finite, it suffices to ask for a transition matrix with positive values in all its positions. In the general case, we need to define carefully what is a meaningful set as it is not possible to list all of them. Fortunately, when the state space Z is a product space between a finite set (Y) and an uncountable subset of $\Re^m$ there is a well know results that help us find an accessible atom in an irreducible chain (for proof, see Proposition 5.1.1 in Meyn and Tweedie.)

Proposition 14 *Suppose that $P^n_\varphi(z, \alpha) > 0$ for all $z \in Z$. Then α is an accessible atom and (Z, P_φ) is a $P_\varphi(\alpha, .)-$irreducible.*

Proposition 14 follows directly from standard results in Meyn and Tweedie ([53]). Note that the reference measure may be different from ψ which is called "maximal". Fortunately, if the chain is

[43] See Meyn and Tweedie ([53]), chapters 5, 8 and 10 for a detailed discussion of the implications of the existence of an atom for the existence of an invariant probability measure.

[44] for example, Stokey, Lucas and Prescott, ([78]), Th. 7.6, p. 184).

[45] For example, see Grandmont and Hildenbrand ([36]).

[46] e.g., see Meyn and Tweedie (([53], proposition 1)

irreducible with respect to some measure, say $P_\varphi(\alpha, .)$, then it can be "expanded" to ψ (e.g, see Meyn and Tweedie, ([53], Proposition 4.2.2).

In order to apply Proposition 14 to our model, the finiteness of Y and the definition of the Markov kernel P will be critical. As we are considering a point, in order to show that $P^n_\varphi(z, \alpha) > 0$, it suffices to find a finite sequence $\{y_0, ..., y_n\}$ such that $\{\alpha\}$ is a solution to equations (52) to (56), the system associated with a binding collateral constraint. We want to associate the atom with an economic crises as this fact will allow us to connect the invariant measure with a sudden stop (SS) or, equivalently, to relate the steady state of the model with the frequency of crises. Typically the literature associates a hit to the collateral constraint with a crises. We are extending these results and connecting the frequency of crises with the stochastic steady state.

The power of an atom for characterizing the behavior of the Markov chain is well-known.[47] However, it is possible to illustrate the effect of an atom in the recurrence structure of the chain, which is critical to define an invariant measure (i.e. a measure μ which satisfy $\mu = \int P_\varphi(z, A)\mu(dz))$. Suppose that the atom is hit for the first time with positive probability in period $\tau_\alpha < \infty$ starting from z_0. Then, it is possible to define a (not necessarily probability) measure μ which gives the expected number of visits to a particular set in $B(Z)$, called it A, before τ_α. Stated differently, $\mu(A)$ gives the sum of the probabilities of hitting A *avoiding* α. Imagine the system in period $\tau_\alpha - 1$ is starting from z_0. Remarkably, when you "forward" μ 1 period (i.e. by applying the Markov operator to it, $\int P_\varphi(z, A)\mu(dz))$ the expected number of visits to A avoiding α is the same as the chain will hit α in period $n = \tau_\alpha$. Thus, μ must not change or equivalently $\mu = \int P_\varphi(z, A)\mu(dz)$. That is, μ is an IM. Provided that $\tau_\alpha < \infty$, it is possible to normalize μ to be a probability measure, which we will call π (this property is called "positivity"). Further, as the accessibility of the atom comes together with the irreducibility of the chain (see proposition 14), it is not surprising that the IM is unique as the chain does not break into different "unconnected islands". Finally, the Krein-Milman theorem insures the ergodicity of the chain provided its uniqueness (see Futia, ([34])).

Now, in order to connect the existence of an invariant measure with the frequency of crises let τ_α be the time when the process hits the collateral constraint. Then, $\mu(A)$ gives the cumulative probability of hitting A *avoiding* a crises. Thus, the frequency of a SS affects the stationary distribution μ. Frequent crises implies more volatility. To understand this relationship for equilibrium paths in the steady state, note that every time the process hits the collateral constraint it reverts to α. Because $P_\varphi(\alpha, A) = \nu(A)$, the value of $z_{\tau_\alpha+1}$ is *independent* of the past, which implies that it loses all the inertia inherited from the Markovian structure of the process. Thus, the equilibrium stochastic dynamics behave unconditionally with respect to the past, increasing its variability.

To relate these facts with the empirical performance of the model, we need a Law of Large numbers. As the measure is ergodic, it is well known that $\sum_{t=0}^{T}(z_t/T) \to E_\mu(z)$ almost everywhere, where $\to$ means T tends to infinite. [48] In other words, the existence of an ergodic measure insures that a sample mean computed by an increasing large time series of simulated data will hit the steady state of the model, represented by the mean $E_\mu(z)$, for a large fraction of possible paths $\{z_t\}_{t=0}^{\infty}$. Thus, the time spells without a crisis shape the long run distribution of the model, affecting its ability to replicate stylized facts. The following theorem proves the existence of an ergodic measure μ for the model described in section 2.

Theorem 15 *There exists an $\varepsilon > 0$ such that $y_{lb} \in (0, \varepsilon)$, a compact set $J_1 \subseteq Z$ with $\Phi : J_1 \times Y \longrightarrow J_1$ and a selection $\varphi \sim \Phi$ such that the process defined by (J_1, P_φ) has an unique ergodic probability measure.*

[47] Meyn and Tweedie mention the importance of an atom for general state space Markov chains relative to countable state space Markov chains (e.g., [53], p96). A discussion of the important of the existence of an atom in the context of the Markov chain theory is outside the scope of this paper, but a systematic discussion of this fact is presented in Meyn and Tweedie (Chapters 8, 10 and 17).

[48] e.g., see Stokey, Lucas and Prescott, ([78]), chapters 11 and 12)

5 Applications

We now apply the results obtained in sections 3 and 4 to characterize the model's SCE described in section 2. It is organized in 3 subsections: i) a comparative analysis of all equilibrium definitions introduced so far and their connection with the degree of memory assumed in each of them. In this section we will focus on the interplay between the different equilibrium concepts and how to use them and their qualitative properties to select multiple equilibria. ii) A characterization of the collateral constraints which focus on multiplicity. iii) A quantitative exploration of short and long run simulations. In this last subsection, we describe the algorithms generated by the Generalized Markov Equilibria (GME) and then use them to compute and simulate an ergodic, a stationary and a non-stationary equilibrium.

5.1 Equilibrium definitions, memory and selection

In this subsection, we split the discussion in 2 parts. First, the interplay between different equilibrium concepts. Sections 2, 3 and 4 all contain its own definition of equilibrium: SCE, RCE and GME respectively. This section ranked them in terms of memory and describes the usefulness of the RCE to construct an ergodic selection of the GME that is efficient from a numerical perspective. Second, we discuss how to use these concepts and their properties to construct a refinement mechanism.

5.1.1 The interplay between SCE, RE and GME

Each definition generates a vector valued function with identical image, $[c_t, d_{t+1}, p_t]$, but different domain. From sections 2, 3 and 4 it is clear that:

- a SCE defines $[c_t, d_{t+1}, p_t]\,(y_0, ..., y_t)$ and has *infinite* memory relative to the minimal state space of *(y,d)*
- a RCE defines $[c, d_+, p]\,(y, d)$ and has *zero memory on the minimal state space (y,d)*
- a GME defines $[c, d_+, p]\,(y, d, m)$ and has *finite memory on the minimal state space (y,d), and zero memory on the enlarged state space (y,d,m)*

From these remarks it is clear that:

$$SCE \supseteq GME \supseteq RCE$$

Even tough we can use the general preferences discussed in section 2 (see assumption 1), to understand the implications of the statements above it is convenient to use a specific utility function and restrict the model using Remark 2 in section 2. Much of the applied literature uses the following functional form for $U(A(c_t))$:

$$U(A(c_t)) = \frac{A(c_t)^{1-\sigma} - 1}{1-\sigma}, A(c_t) = (a(c_t^T)^{1-1/\xi} + (1-a)(c_t^N)^{1-1/\xi})^{\frac{1}{1-1/\xi}} \tag{57}$$

Using this utility function, if we assume $\sigma = 1/\xi = 2$, $U'\left\{A_1(c_t^T)\right\}$ reduces simply to $a(c_t^T)^{-1/\xi}$. Thus, equation (50) becomes

$$[\kappa\{y^T + py^N\} - d_+][(y^T + d_+R^{-1} - d)^{-1/\xi} - E((y_+^T + d_{++}R^{-1} - d_+)^{-1/\xi})] = 0, \tag{58}$$

where we have used the fact that m is the envelope of the dynamic programming problem, $m = V' = a(c_t^T)^{-1/\xi}$. Note here the relevance of the qualitative results in section 3.1: the envelope is a well defined derivative and it is deeply connected with the dual representation of the equilibrium. In this sense, we are using the primal and the dual version of the households optimization problem to construct the GME.

We now turn to the relationship between the RCE and the GME. After applying the forward operator, replace m_+ on the right hand of equation (50). Let define g to be the equilibrium policy function for d_+ in a RCE. In particular, $d_+ = g(y,d) = [A^*(C^T(y,d) + d - y^T)]R$ where $C^T \in \{\wedge\Psi^*, \vee\Psi^*\}$ according to theorem 9. Similarly $d_{++} = g(y_+, g(y,d))$ and $p(y,d)$ satisfies equation (24) at the RCE. Note remarkably that, depending on the initial condition $\{0, c_{max}\}$, we will respectively converge to $\{\wedge\Psi^*, \vee\Psi^*\}$ with $\wedge\Psi^* \leq \vee\Psi^*$. That is, if agents are pessimistic (i.e., start the iterative process in theorem 9 at $C_0^T = 0$), they will end up with a low equilibrium consumption - debt pair. The contrary happens if they are optimistic (i.e., start the iterative process in theorem 9 at $C_0^T = c_{max}$). That is, we found 2 different equilibria connected to the dynamics in the equilibrium Euler equation. We refer to this source of multiplicity as *dynamic pecuniary externality.* Note that this source of multiplicity is present given any possible number of roots in the collateral constraint, expressed in equation (41). If this last equation has more than 1 solution, for any given $C^T \in \{\wedge\Psi^*, \vee\Psi^*\}$, we obtain another source of multiplicity that we call *static pecuniary externality.* Thus, equation (58) becomes:

$$[\kappa\{y^T + p(y,d)y^N\} - g(y,d)][(y^T + g(y,d)R^{-1} - d)^{-1/\xi} - E((y_+^T + g(y_+, g(y,d))R^{-1} - g(y,d))^{-1/\xi})] = 0 \quad (59)$$

In section 4 we showed that the GME has an ergodic selector. However, we were silent as regards ergodicity in section 3, when we characterized the RCE. Equation (59) explains why: we can't show the ergodicity of an RCE simply because there is not enough memory on the equilibrium definition or equivalently because there are not enough degrees of freedom to model the future. By definition, the GME has an additional state variable, m. Note that picking m as a state variable, because of its definition and the existence of a well defined envelope, is equivalent to set d_+ as an additional state variable. In this sense, the qualitative properties of the dynamic programming program described in section 3.1 not only guarantee the existence of a standard envelope for any RCE in the presence of multiple equilibria, but also are essential to pick the appropriate additional state variable in the GME to construct an ergodic selector. Thus, under a GME equation (58) becomes:

$$[\kappa\{y^T + p(y,d,d_+)y^N\} - d_+][(y^T + d_+R^{-1} - d)^{-1/\xi} - E((y_+^T + \varphi(y_+, y, d, d_+)R^{-1} - d_+)^{-1/\xi})] = 0, \quad (60)$$

where φ is a selection from the equilibrium correspondence used to define a GME in section 4. Equation (60) shows an immediate implication of expanding the state space: d_{++} depends now on an additional variable, adding memory to the cognitive structure of agents. In equation (59), after setting d to a constant value, d_{++} depend only on exogenous shocks. It is clear from (60) that d_{++} is not restricted to satisfy $d_{++} = g(y_+, g(y,d))$. Thus, we have $SCE \supseteq GME \supseteq RCE$ as the SCE does not even has the stationarity requirements of the GME.

We now turn to the relationship between the RCE and the ergodic GME (EGME) (and in particular, discuss the role of the uniqueness of the unconstrained problem in the RCE on the characterization of EGME). Section 4 showed that this additional degree of freedom is sufficient to obtain an ergodic representation. To construct an EGME we use the qualitative properties in the unconstrained problem defined in section 3, equation (37). In particular, this expression is increasing in d for a constant y. Thus, we can construct a positive probability path such that the collateral constraint is binding in finite time, which in turn allows us to return to the atom if we set $y = y_{lb}$.

Note that the notion of dynamic pecuniary externality, which we defined above, is absent in the GME. However, as in the RCE, the existence of a static pecuniary externality associated with equilibrium collateral constraint arises when aggregators $A(c)$ are such that the mapping $Z^*_{cc}(x, x, s^e, \kappa/R)$ in the definition of $X^*(s^e)$ in equation (47) has many roots. In this sense, when Assumption 3 is imposed, for the RCE the model cannot display dynamic pecuniary externality (which follows, in essence, from Theorem 11 in section 3). Moreover, as Lemma 7 showed, the unconstrained policy function is unique conditioned on the parameterization of the collateral constraint with $C^T(S)$. So if there is no static pecuniary externality, as occurs under Assumption 3 in section 3, in our RCE operator equation we set the collateral constrained consumption tomorrow to be $\inf\{c(d,y), c_c^{T*}(d,y)\}$; where $c_c^{T*}(d,y)$ is the pointwise unique solution for the collateral constraint in equation (71), which in effect eliminates also any period dynamic pecuniary externality. This implies under Assumption 3 on the consumption aggregator $A(c)$, Lemma 7 directly implies Theorem 11, and therefore iterations on our RCE fixed point operator from *any* initial $c \in \mathbf{C}^p_{++} = \{c \in \mathbf{C}^p | c(d,y) > 0\}$ in Theorem 9 converges to the unique RCE in the unconstrained states, and therefore the RCE tradables is $c^*(d,y) = \inf\{c^{T*}(d,y), c_c^{T*}(d,y)\}$.

Now, under the conditions discussed in the previous paragraph, there is an EGME for each RCE and they are completely characterized by the roots of the collateral constraint. Of course, the dynamics of an EGME and of the RCE will be different as the latter does not revert to the atom anytime the model hits the collateral constraint. The RCE is not ergodic because of the lack of memory.

The discussion above shows that the static pecuniary externality is present in both, RCE and EGME. To construct a stationary equilibrium, we must select one of the multiple roots together with the initial conditions of the dynamical system induced by any of these equilibria. This can be interpreted as an additional i.d.d. variable (i.e., root selection) that must be constant to preserve stationarity. Section 5.2 shows how to handle the static pecuniary externality in the EGME / RCE and provides a restriction on the preferences that eliminate it. Section 5.3 will show that the EGME can be computed efficiently. Thus, this paper contains a toolkit to deal with models of financial crises in small open economies composed by: i) a definition of equilibrium, the GME, ii) a steady state, the ergodic invariant measure discussed in section 4, and iii) an algorithm based on some qualitative properties of the RCE, which will be described in section 5.3.

5.1.2 How to handle multiple equilibria

We now summarize the discussion in the preceding section and provide a simple guide on how to handle multiplicities.

RCE

- Static pecuniary externality. Choose one of the roots of the collateral constraint in (41). Section 5.2 will show how to select among 2 possible roots for a standard utility function in the literature and will provide sufficient conditions for a unique root in the collateral constraint.

- Dynamic pecuniary externality. Initialize the first step operator (42) with the upper bound in the space of candidates c_{max} and find a first step fixed point, which is unique due to lemma 7. Then, move to the second step. Choose an initial condition $\{0, c_{max}\}$ and using the operator (46) to update the equilibrium policy function. Theorem 9 guarantees monotonic convergence to $\{\wedge\Psi^*, \vee\Psi^*\}$. The procedure is constructive and thus contains a selection mechanism for the multiplicity inherited from the dynamic pecuniary externality.

EGME

- Expanded state space. Add an additional state variable m with respect to the minimal state space (y, d). In the state space (y, d, m) equations (52)-(56) define a determined system using the equilibrium correspondence Φ in definition 12.

- Suitable additional state variable. Due to equations (21) and (22), the envelope is well defined. Then, we can use equation (60) to define a selection $\varphi \in \Phi$ that is constructed as follows: i) when the collateral is not binding, use the unconstrained policy function in (37), recovering p from (24).

ii) When the constraint is binding, say at y, d, d_+, hit the atom (y_{lb}, d_*) using equation (60) where $d_+ = R(c(y_{lb}, d_*) - y_{lb} + d_*)$ and $c(y_{lb}, d_*)$ is given by equation (32). Notice that at the atom both equation (56) and the collateral constraint are satisfied with equality but at y, d, d_+ we may have a strict inequality in either one or both equations.

- Numerical efficiency and ergodicity. Lemma 7 (a) and (b) implies that the GME is easily implementable: we can use the unconstrained problem until we hit the collateral constraint, then one of the roots of this constraint defines the atom. As in the RCE, we must choose one of these roots for any time period to stationarize the process and construct a selection φ with $d_{++} = \varphi(y_+, y_{lb}, R(c(y_{lb}, d_*) - y_{lb} + d_*), d_*)$ when the collateral is binding and $d_{++} = \varphi(y_+, y, R(c(y, d) - y + d), d)$ at any other time. Theorem 15 implies that this process is ergodic.

It is clear that the static pecuniary externality is crucial for both types of recursive equilibria. In the next section we carefully deal with this type of externality.

5.2 Multiplicity of equilibrium and static pecuniary externality

In this section we first discuss the connection between multiplicity and static pecuniary externalities. Then, we provide a sufficient condition to eliminate the former and thus the latter. It turns out that the intra-temporal elasticity of substitution is a key element behind multiplicity and static externalities. By controlling this elasticity not only we eliminate both these features of the model, but also in some cases the presence of a spiralized crisis often referred as *Fisherian deflation* (see Bianchi ([15]) among others). Thus, severe balance of payment crises and multiple equilibria are deeply connected.

5.2.1 Multiple equilibria under static pecuniary externalities

From the discussion above, it is clear that the RCE and the EGME generate multiple selections. While the dynamic pecuniary externality is only present in the latter, the static pecuniary externality affects both of them. This source of multiplicity, frequently studied in the literature (see Schmitt-Grohé and Uribe ([74])), is concerned with the possibility of having more than 1 solution to equation (52) when the collateral constraint is binding. Note that this equation defines a stationary system in debt today d for any level of tradable income y^T. Thus, to characterize it, the bounds on marginal utility generated by assumption 1 are not required. As this section is concerned with this type of multiple equilibrium, we will use standard CES preferences as, for instance, in Bianchi ([15]). Given a uniform upper bound on debt, for the RCE and the GME, this assumption can be imposed to make our results comparable with the literature (see remark 2 in section 2).

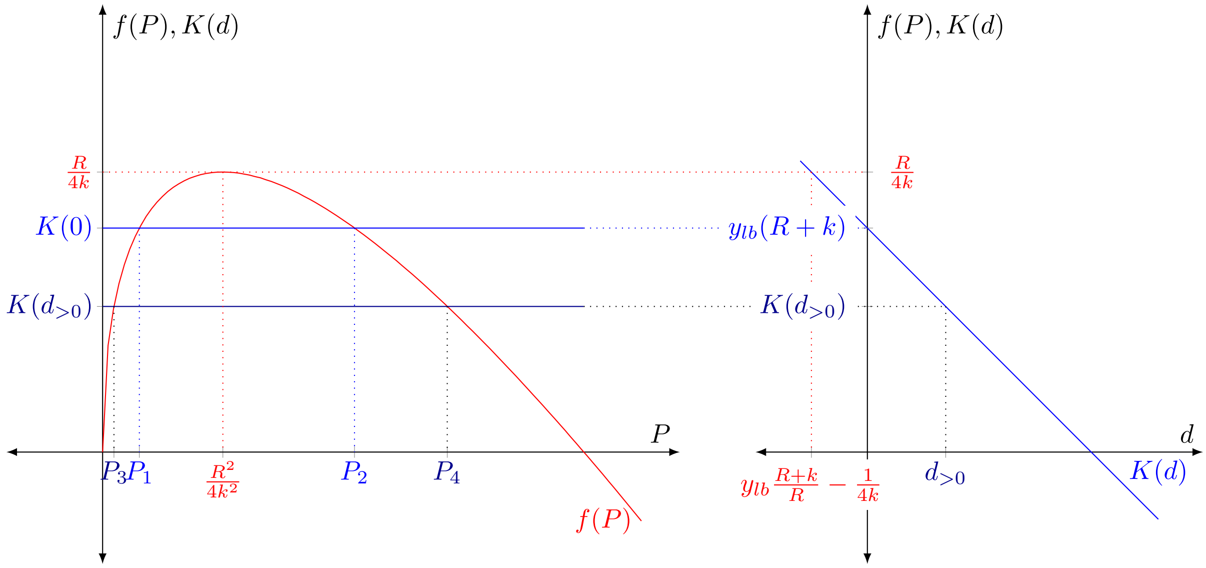
Let $U(A(x)) = (A^{1-\sigma} - 1)/(1-\sigma)$, $A(c) = (a(c^T)^{1-1/\xi} + (1-a)(c^N)^{1-1/\xi})^{1/(1-1/\xi)}$ with $\sigma = 1/\xi = 2$ and $a = 1/2$. We will assume that Z is compact. Then, equation (52) becomes:

$$f(P) \equiv P^{1/2} - \kappa P y^N = y_{lb}/y^N + \kappa y_{lb} - d/y^N \equiv K(d), \tag{61}$$

where we have set $y^T = y_{lb}$ and $(1-a)/a = Ry^N = 1$. As (61) for the GME is the analogous of (41) for the RCE when C^T is at its stationary equilibrium value, we can characterize the static pecuniary externality in both equilibrium types using this equation. The left hand side of (61) is a function of P and the right hand side of d. Let $f(P)$ and $K(d)$ be the former and the latter respectively. As we have assumed $Ry^N = 1$, f is increasing for $0 < P < R^2/(4\kappa^2)$ and decreasing otherwise (for $P > 0$, of course). Further, $f(R^2/(4\kappa^2)) = R/2\kappa = K(y_{lb}(1 + \kappa R) - (1/2\kappa))$ and $K(0) = y_{lb}(R + \kappa)$. Figure 1 illustrates equation (61) for the described parametrization with $R^2/(4\kappa^2) \equiv P^*$.

The K locus depends on d. The f locus depends on P, that is depicted in the "x-axis". The $K(0)$ line represents the smallest possible value for d in the constrained regime (i.e. $d = 0$). Between $K(0)$

Figure 1: Equation (61), $y^T = y_{lb}$



The K locus depends on d, which is not depicted in the figure above. The f locus depends on P in the "x-axis". The $K(0)$ line represents the smallest possible value for d in the constrined regime (i.e. $d = 0$). Between $K(0)$ and $R/2\kappa$ the regime is not collateral constrained. Below $K(0)$ and over the locus formed by f lie all the candidate pairs (d, P) for the constrained regime.

and $R/4\kappa$ the regime is not collateral constrained. Below $K(0)$ and over the locus formed by f lie all the candidate pairs (d, P) for the constrained regime.

Note that for $d = 0$ there 2 possible exchange rate levels, $P1$ and $P2$, and a change in d with $d > 0$ can either increase or decrease P. This is depicted in points $P3$ and $P4$ in the same figure. Moreover, an increase in y^T implies that the $K(0)$ locus must jump upwards while the $f(P)$ locus remains constant as it is independent of tradable output by construction. Figure 2 illustrates this situation. Note that the collateral constraint doesn't bind when the agent saves (i.e. $d < 0$) as endowments and prices are positive. Thus, after the depicted increase in y^T, the region of possible multiple prices for a positive level of debt now includes the whole f locus.

Figures 1 and 2 illustrate the implications of the stochastic structure in one of the sources of multiplicity of equilibrium: *as we increase the shock level from y_{lb} to y^T there is an increase in the admissible (positive) debt levels which can generate multiple equilibria.* This fact follows immediately from the definition of K.

The following claim states the existence of multiple equilibrium for a sufficiently rich set of shocks. See the supplementary material for this subsection in the online appendix for a detailed discussion.

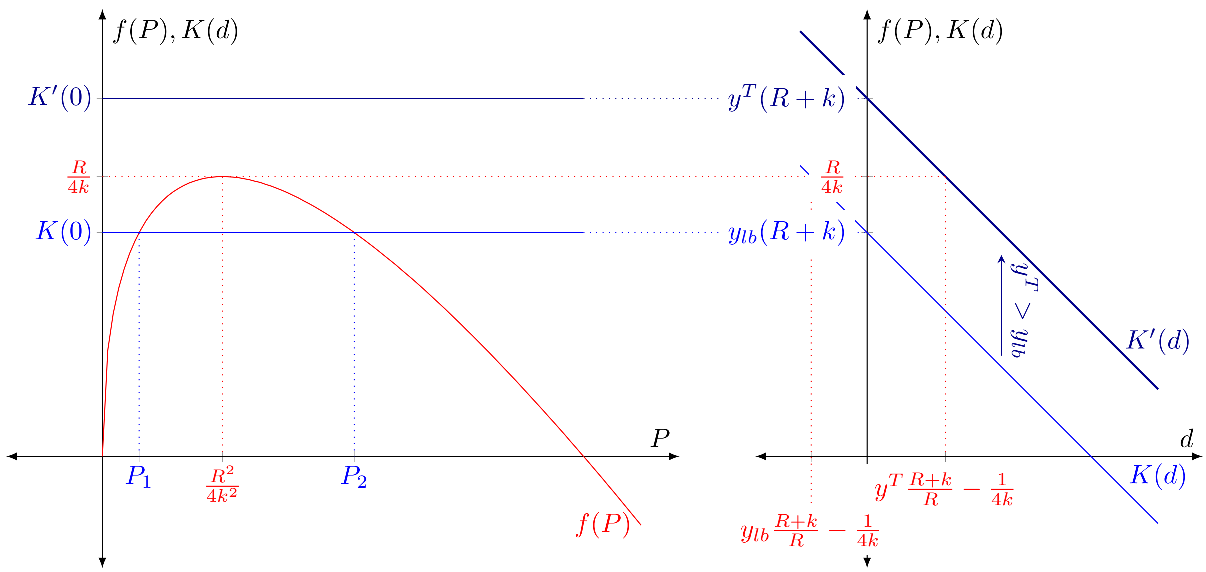
Assume the equilibrium set is compact. Then, if the set containing exogenous shocks Y has at least 3 elements, then the system of equations formed by (52) *to* (56) *has 2 solutions for at least 2 different elements in Y (i.e. the model has multiple equilibrium).*

5.2.2 Static pecuniary externality and intratemporal elasticity of substitution

In this subsection we present 2 examples of different intra-temporal preferences to study the interaction between static pecuniary externalities and multiple equilibria. We find that, even though the absence of a static pecuniary externality is sufficient to obtain a unique equilibrium, it is not necessary.

Just to keep the paper self-contained, let us define the intratemporal elasticity of substitution:

Figure 2: Equation (61), $y^T > y_{lb}$



An increase in y^T shifts the K line. Thus, the whole locus f contains all the candidate pairs (d, P) for the constrained regime.

$$\zeta_{IES} \equiv \partial ln(c^T/c^N)/\partial ln(MRS(c^N, c^T)) = \partial ln(c^T/c^N)/\partial ln(p).$$

If $U(A(x)) = (A^{1-\sigma} - 1)/(1-\sigma)$, $A(c) = (a(c^T)^{1-1/\xi} + (1-a)(c^N)^{1-1/\xi})^{1/(1-1/\xi)}$, $\zeta_{IES} = \xi$.

The discussion in the preceding subsection implies that as long $\xi = 0.5$, we will generally have: i) static pecuniary externalities, ii) multiple equilibria, iii) a spiralized (i.e., observed in at least 2 consecutive periods) balance of payment crises characterized by deleverage and a real depreciation (i.e., a reduction in both d and p). The critical aspect behind i) and iii), ii) was extensively covered in the previous subsection, is captured by the collateral constraint in equilibrium: $d_+ = \kappa(y^N p(c^T) + y^T)$, where the dependence of p on c^T follows from equation (52). A balance of payment crisis implies a reduction in c^T, which pushes down debt through the collateral constraint and a depreciation (i.e., a reduction in p as it is increasing in c^T). The supplementary appendix contains details about these dynamics.

Now consider imposing a quasi-linear intra-temporal structure of preferences. That is, assume that:

$$\tilde{A}(c) = c^T + \frac{(c^N)^{1-1/\xi}}{1-1/\xi}, \tag{62}$$

where we assumed that $a = (1-a)$. Under equation (62), p, characterized by equation (52), becomes $p = (y^N)^{-1/\xi}$ and a binding collateral constraint implies $d_+ = \kappa((y^N)^{1-1/\xi} + y^T)$ which not only has 1 root, but also it breaks the spiralized recession as it rules out the static pecuniary externality (i.e., the dependence of p on c^T).

The intuition behind the uniqueness is straight forward: the marginal rate of substitution, equation (52), is independent[49] of c^T. As GDP expressed in tradables, which equals $py^N + y^T$, is exogenous and (intra-temporally) tradable consumption only responds to changes in income due to the quasi-linear structure of preferences, the intra-temporal behavior of tradable consumption is exogenous. Thus, there is neither static pecuniary externality nor spriralized recession. Of course, inter-temporaly tradable consumption is driven by the standard consumption smoothing channel which is captured by (37). Lemma

[49]With quasi-linear preferences the elasticity of substitution is not constant but is 0 for compensated price changes

7 implies that the policy function driven the unconstrained inter-temporal channel $d(d, y^T)$ is unique. Moreover, as the collateral constraint only has one root, uniqueness is guaranteed. Finally, as $d_+ = min\{\kappa((y^N)^{1-1/\xi} + y^T), d(d, y^T)\}$, the equilibrium is continuous which in turn guarantees its ergodicity and stationarity by standard results. Thus, *with quasi-linear preferences we don't have static pecuniary externalities and the equilibrium is unique* (i.e., the absence of static pecuniary externalities is sufficient for uniqueness). We now turn to necessity.

Assume that $\bar{A}(c) = ln(c^T) + ln(c^N)$. That is, $\zeta_{IES} = \xi = 1$ and $p = c^T/c^N$. In this case, we have:

$$p = \frac{(1+\kappa)y^T - d}{(1-\kappa)y^N} \tag{63}$$

Equation (63) implies the necessity of 2 additional restrictions: i) $p \geq 0$ and ii) $py^N + y^T \geq 0$. The former implies $(1+\kappa)y^T \geq d$ and the latter $2y^T \geq d$. By assuming $0 < \kappa \leq 1$, we can get rid of ii) as i) will bind first. Now the collateral constraint is given by: $d_+ \leq \kappa(1-\kappa)^{-1}[2y^T - d]$. Thus, we have the following restriction:

$$d_+ = min\{d(y^T, d), \kappa(1-\kappa)^{-1}[2y^T - d], (1+\kappa)y^T\} \tag{64}$$

Equation (64) implies that if $0.5 < \kappa \leq 1$, then $d_+ = min\{d(y^T, d), (1+\kappa)y^T\}$, which implies uniqueness, continuity and the absence of pecuniary externalities. However, if $0 < \kappa \leq 0.5$, we have $d_+ = min\{d(y^T, d), \kappa(1-\kappa)^{-1}[2y^T - d]$, which implies uniqueness but the model displays pecuniary externalities and spiralized recessions. To verify this claim we must have:

$$d_{++} = \kappa(1-\kappa)^{-1}[2y^T - (\kappa(1-\kappa)^{-1}[2y^T - d])] \tag{65}$$

$$\kappa(1-\kappa)^{-1}[2y^T - (\kappa(1-\kappa)^{-1}[2y^T - d])] < \kappa(1-\kappa)^{-1}[2y^T - d] \tag{66}$$

Equation (65) implies that the collateral constraint is binding for 2 consecutive periods and equation (66) that there is deleverage. While the latter follows directly if $0 < d < y^T$, the former requires a more subtle argument. Note that to verify equation (65) we need to have $d(y^T, x) > \kappa(1-\kappa)^{-1}[2y^T - x]$, where x is a potential value for debt, for at least 1 level tradable output y^T. As $d(y^T, x)$ follows from a standard savings problem, we must have $d(y_{ub}, 0) < 0$. Let K_2 be the compact set containing d according to lemma 3 and $K_2^{lb} < 0$ its lower bound. As $d(y, .)$ is increasing in x for any y, we have $d(y_{ub}, K_2^{lb}) < 0$. Then, $\kappa(1-\kappa)^{-1}[2y_{ub} - x]$ is linear and decreasing in x with $\kappa(1-\kappa)^{-1}[2y_{ub} - K_2^{lb}] > 0$. Then, there exist x^* with $d(y_{ub}, x^*) = \kappa(1-\kappa)^{-1}[2y_{ub} - x^*]$. The last 2 inequalities imply that we have: $d(y_{ub}, x) > \kappa(1-\kappa)^{-1}[2y_{ub} - x]$ for $x \in (x^*, K_2^{ub}]$ as desired where K_2^{ub} is the upper bound of K_2. Thus, *we show that with log preferences, we have a unique continuous equilibrium and static pecuniary externalities. That is, uniqueness is not equivalent to the absence of static pecuniary externalities.*

5.3 Empirical Procedure, Algorithms and Simulations

We now turn to the quantitative implications of the results presented in sections 3 and 4. Taking into account the lack of a closed form solution, we must take care of the numerical approximation of the model presented in section 2.

We first show how to compute the ergodic selection in the GME. It can be used to simulate a recurrent, and thus ergodic, behavior as the stochastic paths visit the atom in a crisis. Following the theoretical results in section 4, each ergodic selection has a unique invariant measure. Then we solve the model for a parameter set borrowed from the empirical literature and compute the effects of a change in the interest rate in the long run of the model.

We investigate numerically the difference between a ergodic, a stationary and a non-stationary equilibria. The discussion in section 5.1.1 shows that it is possible to remove the ergodic component of any selection in a GME simply by visiting a different point every time the collateral constraint binds. However, this selection is still time invariant and thus, stationarity We compute the difference between simulations generated by an ergodic and a stationary GME equilibrium. We found that ergodic simulations generate smoother consumption paths or, equivalently, agents are less financially constrained. Finally, we compute a non-stationary GME. As it is expected, this equilibrium can generate large fluctuations in macroeconomic fundamentals (i.e. current account) using standard preferences and with the same shock structure. That is, simply by changing the definition of equilibrium, if we are willing to give up the long run performance of the model, the same theoretical structure is capable of generating a great range of balance of payments crises.

5.3.1 Numerical procedure

We describe how to compute an ergodic selection and use it to simulate the long (a path of length N) and short run (of length $T < N$) behavior of the model. The online appendix contains additional details.

Let (d, d_+) denote the debt levels observed today and tomorrow. The presence of the collateral constraint implies that $d_+ \neq d(d, y)$, where as before $d(.,.)$ denotes the policy function in the unconstrained regime for the RCE. Thus, the ergodic selection of the GME φ depends on $\varphi(d, y, y_+)$ for every $(d, y, y_+) \in K_2 \times Y \times Y$ if the collateral does not bind and on $\varphi(d_+, d, y, y_+)$ for every $(d_+, d, y, y_+) \in K_2 \times K_2 \times Y \times Y$ if it binds. The connection between the RCE and the GME described in section 5.1.1 guarantee the existence of a stationary structure for the GME given by equations (52)-(56) and $d(.,.)$. This fact implies that the GME is computationally efficient. That is, once $d(.,.)$ is available, it can be computed fast.

The algorithm in the online appendix, called GME ergodic algorithm, generates a sequence $\{p_t, d_{t+1}\}_{t=0}^T$ which *depends* on $p_{T+1}(y_{T+1})$ and $d_{T+1}(y_T)$ for a given point in the set of deep parameters Θ. This variables are pinned down by picking an ergodic selection for the GME. The results in section 5.1.1 implies that we need to constraint the paths generated by the GME using the policy function of the RCE. However, this is not necessarily the case if we only need a stationary GME. That is, in this last case, we may allow $d_+ \neq d(d, y)$ even if the collateral does not bind. Thus, we are adding memory to the selection as d_+ may not depend on (d, y). Contrarily, any sequence generated from the minimal state space algorithm depends *only* on the point in Θ and the draw from the stochastic process which generates tradable output, for a given d_0, y_0. That is, $\varphi \in \Phi$ *does not necessarily satisfied* $d_{T+2}(y_{T+1}) = d(d(d_T, y_T), y_{T+1})$ as it is the case for the minimal state space algorithm.

The online appendix also contains the numerical procedure to compute a stationary GME. The function which maps $[y_t, p_t, d_t] \longmapsto [y_{t+1}, p_{t+1}, d_{t+1}]$ for each $y_{t+1} \in Y$, which defines the Markov kernel in the GME, *does not necessarily satisfy* the structural properties required to prove the existence of a RCE. Thus, the path $\{p_t, d_{t+1}\}_{t=0}^T$ is more flexible as the transition function in the GME can be computed pointwise as in the SCE. This is the *numerical implication of expanding the memory in the recursive equilibrium as the transition function φ is computed exactly as in the SCE* (i.e., pointwise for each element in the draw from (Y, q)).

The online appendix also describes a non-stationary GME Algorithm. In this case the sequence $\{p_t, d_{t+1}\}_{t=0}^T$ *depends* on the histories of the form $p_y(y^t)$, $d_t(y^{t-1})$ with $y^t = y_0, ..., y_t$. Thus there is a trade off: *we gain flexibility with respect to the stationary / ergodic GME in order to incorporate more "memory" from the SCE but in return we can not claim that these paths are connected with the steady state of the model* and that they are independent of time.

5.3.2 Results

We now solve the model. The table below contains the parameters, borrowed from Pierri, et. al. ([63]).

Table 1: Parameters

Parameter	κ	β	σ	ξ	a	z_l	z_h	$p(z_l)$	R	R'
Value	0.3	0.99	2.0	0.5	0.5	0.5	1.5	0.2	1.05	1.025

We now show the results of simulating the ergodic and the stationary process. We present the effects of a reduction in the interest rates in both cases for consumption (Table 2) and debt (Table 3).

Table 2: Summary of Simulations Statistics for Consumption

Statistics	Mean(R)	STD(R)	Mean(R')	STD(R')
Ergodic	1.0497	0.450	1.0501	0.453
Stationary	1.01	0.52	1.04	0.53

Table 3: Summary of Simulations Statistics for Debt

Statistics	Mean(R)	STD(R)	Mean(R')	STD(R')
Ergodic	0.230	0.086	0.235	0.094
Stationary	0.27	0.06	0.29	0.07

The statistics are reported at distinct truncation levels as there are some cases for which 2 decimal positions are not enough to differentiate between simulations. A reduction in the interest rate generates the expected changes in ergodic and stationary simulations: an increase in consumption and thus a reduction in the savings rate which implies more debt. However, there are at least 2 connected differences: stationary simulations overestimate i) the elasticity of average consumption and debt with respect to the interest rate, ii) the volatility of consumption for the same interest rate, which implies that debt is less volatile. For the first fact, the intuition goes as follows: as the atom is not directly affected by the change in the interest rate, only through its effect on the unconstrained policy function in the RCE, and ergodic simulations are generated as a sequence of recurrent sets which has a regeneration point in the atom, average observed endogenous variables are not severely affected. For the second fact, note that in the ergodic simulation debt is regenerated to a very low level as by construction the atom is defined to hit the collateral constraint with equality. Thus, de-leveraging is more significant in an ergodic crises, which implies that the economy has more time to accumulate debt and to smooth consumption.

The take away point from the above results is related to the invariance of the atom with respect to parameter changes. The most direct way to change the ergodic distribution is to affect the regeneration point. In this case, numerically, the atom does not change significantly after the interest rate shock even tough the unconstrained policy function and the value of the collateral are both affected. In particular, the atom is given, as described in the appendix, by:

$$d(d_*, y_{lb}; R) = \kappa \left[y_{lb} + y^N \left(\frac{A_2(y^N)}{A_1(y_{lb} + (d(d_*, y_{lb}; R)/R) - d_*)} \right) \right]$$

After the change in the interest rate both the left and right hand side of the equation above rises as $d(d_*, y_{lb}; R)$ goes up and $d(d_*, y_{lb}; R)/R$ goes down. Thus, the change in d_* is not significant. Finally, we present a non-stationary simulation in the table below.

Table 4 reflects the flexibility contained in the model. By changing selections, the SCE is able to replicate a sharp reversion in the current account, as it is frequently observed in data, without requiring a change in the parameters (i.e. a reduction of κ or y_{lb}).

Table 4: Non Stationary Crises

Non Stationary Simulations	$t-1$	t	$t+1$
Current Account / GDP	-10%	-8%	$+5\%$

6 Extensions and Concluding Remarks

We conclude the paper with a brief discussion of how the results of this paper can be extended to more general versions of this model. In particular, we discuss the case of continuous shock space and nonhomothetic preferences. Relative to the former issue of continuous shock spaces, as well as non-homethetic preferences, the results on the existence, computation, and characterization of RCE in section 3 change very little. On the other hand, the results for section 2 on the existence of more general SCE than RCE with continuous shock spaces requires significant changes in constructions, and this topic is beyond the scope of the existing paper.

In particular, for the case of general continuous shocks spaces and Markov shocks, the results in Section 3 on RCE generalize directly. First, with continuous shock spaces, measurability issues are now not trivial. They also change the order completeness properties under pointwise partial orders of the spaces $\mathbf{C}^p$ (resp., $\mathbf{C}^*$) in lemma 6 (resp., lemma 8). In particular, both spaces are now only $\sigma-$complete lattices (i.e., complete only relative to countable subsets). As our RCE existence mapping in Theorem 9 is based on order continuous operators, then one change in the results is that in Theorem 9, we the set of RCE now is only countable chain complete. (e.g., see Balbus, Reffett, and Wozny [10], Theorem 7.1). The iterative monotone comparative statics results for RCE do not change as the theorems applied from Balbus et al. ([8], [9]) only require the space $\mathbf{C}^*$ in lemma 8 to be $\sigma-$complete lattice. So all the results of section 3 on RCE can be directly extended to the case of continuous shock spaces. As RCE induce SCE, we also know that SCE exist.

Now, the complication with continuous shock spaces (and especially Markov shocks) is extending the more general existence results of SCE beyond the set of RCE, and most importantly extending the ergodicity results in Sections 4 and 5 to the Markov shock case. As mentioned earlier in the paper, RCE are only short-memory SCE, and the stochastic equilibrium dynamics associated with such RCE are very simple relative to more general forms of SCE. For more continuous shock spaces, the existence of SCE as proved in Theorem 4 is requires a more delicate construction (e.g., see our companion paper in Pierri and Reffett ([64]). In principle, this can be done. Unfortunately, the ergodicity results in section 4 and 5 using GME representations are not so easily extended to the Markov shock case. So this will be a subject for future work.

Additionally, when constructing the set of RCE, as we mentioned in section 3, we imposed Assumption 2, which imposes a standard Inada condition on the marginal utility of consumption in tradeables. This is not assumed in Assumption 1 that is used to prove the existence of SCE (where we place an upper bound on consumption as $c^T \to 0$ (i.e., bounds consumptions uniformly on the interior of the consumption set). In Assumption 1, we guarantee this uniform inferiority of consumption by imposing a non-homothetic parameter in the assumption of the aggregator $A(c)$. As we argue in the paper (and show for our numerical results), one can make the "non-homothetic" term very small so in the end, it does not matter for our results relative to using a standard Armington aggregator. But there is a complication for computing the least RCE when Assumption 2 is not present. We need to guarantee the existence of a "lower subsolution" in Theorem 9 to compute a least RCE. Guaranteeing the existence of this element requires finding a "sufficiently high" marginal utility of tradeables consumption near 0. This greatly complicates the characterization of least RCE. Nothing changes for the characterization of greatest RCE. But under Assumption 1, RCE still exist.

To see how that would be case, simply note that if we have the preferences that only satisfy assumption 1, we can always pick the non-homothetic parameter used to guarantee sttrictly positive consumption to be arbitrarily small so that consumption of tradeables (in a RCE) stays strictly positive (and an Inada condition for strictly positive consumption that guarantees the existence of a roots to define our RCE operator in the "unconstrained states" in equation (40). So all the results on existence and characterization of RCE go through under either the assumptions that guarantee uniformly interior consumptions (as in Assumption 1), or the existence of strictly positive consumption in all states (as done using Assumption 2)

One other question we are considering per future work are versions of the sudden stops model with production (e.g., as in Benigno et al ([12]). Here the authors allow for endogenous labor supply that is used by firms in the economy producing both tradeables and non-tradeables consumption good (but there is a fixed capital stock). If the preferences are Greenwood-Huffman-Hercowitz (GHH) preferences, nothing in our arguments really change. If preferences over leisure are more general, the arguments in this paper have to be changes a great deal, but still the methods of the paper can be extended to such economies. See Pierri and Reffett ([64]).

Finally, another interesting and important variation of this model is the case when collateral constraints on debt are based on *future* wealth of the household, not the current wealth. The importance of these issues have been discussed in Ottonello, Perez, and Varraso ([60]), and in a related context in Brooks and Dovis ([19]). When collateral constraints in sudden stops models are based on future wealth, it turns out that the multiplicities of dynamic equilibria question can become very subtle. For example, it is not clear at all that our uniqueness of RCE result will extend to this case. Further, even to develop sufficient conditions for the existence of either SCE and/or RCE becomes substantially more demanding. Finally, the stochastic dynamics of the model appear to be much more complicated to characterize, so it is not clear how to extend the ergodicity results for using GME representations of SCE (even if they exist). We are considering this important case in our future work.

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Appendix

Proofs for section 2

We now turn to the characterization of the SCE. In order to get the paper self-contained, we write the primal version of the first order conditions which will be useful to prove the long run properties of the equilibrium. We keep the notation $c_1 \equiv c^T, c_2 \equiv c^N$ in line with Assumption 1.
For any given sequence of prices $p = \{p_t\}_{t=0}^{\infty}$, from the first order conditions in (7-10), we can obtain the following two expressions critical in characterizing the stochastic structure of binding collateral constraints:

$$[c_2(y^t)][\{\beta^t U'(A(c(y^t))\}\{-A_1(c_1(y^t))p(y^t) + A_2(c_2(y^t))\}] = 0, \quad y^t - a.e \tag{67}$$

$$\begin{aligned} &[\kappa\{y^t + p(y^t)y^N\} - d(y^t)][U'(A(c(y^t))A_1(c_1(y^t)) \\ &-E_t(U'(A(c(y^t y_{t+1}))A_1(c_1(y^t y_{t+1}))] = 0, \quad y^t - a.e \end{aligned} \tag{68}$$

where E_t can be obtained using the $y_t th$ row of the transition matrix if the cardinality of this shock set Y is finite, or by integrating using the density associated with χ evaluated at y_t if Y is an uncountable set. In the i.i.d case, we have $E_t = E$.
We make a few remarks on (67) and (68). First, the characterization of optimal solutions uses the primal formulation of the problem, and hence is written in terms of the "complementary slackness" version of the Karush-Kuhn-Tucker (KKT) equations. In particular, in either equation (67) or (68), the first bracket contains the inequality constraint, $c_2 \geq 0$ in equation (67) or $\kappa\{y^t + p(y^t)y^N\} - d(y^t) \geq 0$ in equation (68), and the second bracket consists of the derivative of the objective function with respect to the control, $c_2(y^t)$ in equation (67) and $d(y^t)$ in (68). Additionally, note we have eliminated the control, c_1, and the restriction in equation (2) from the KKT system. Formally, $c_1(y^t)$ must be replaced with $-p(y^t)c_2(y^t) - d(y^{t-1}) + y^t + p(y^t)y^N + \frac{d(y^t)}{R}$. This last issue is relevant for the dual formulation of the problem as it allows us to avoid dealing with the Lagrange multiplier associated with (2). It turns out that the dual representation is more difficult to characterize in terms of its dynamic behavior when compared with the primal version.
Now, in order to close the characterization of the model, we need a terminal condition on the right hand side of the Euler equation $\beta^t E_t(U'(A(c(y^t y_{t+1}))A_1(c_1(y^t y_{t+1}))$, which results after iterating equation (68) (see Constantinides and Duffie ([25]) for a discussion). Under assumptions 1-a.i, 1-a.iii, 1-e, and 1-f, this requirement will be satisfied. The relevance of these assumptions, their relationship with the restriction we place on βR and the sufficient conditions for the compactness needed to obtain the existence of SCE will be proved in lemma 1, stated in the body of the paper. Note that the results below are necessary conditions. The associated sufficient conditions for existence will be proved in theorem 1.

Proof of Lemma 3

Proof. I) Let $Y = [y_{min}, y_{max}]$. From assumption 1-g), lemma 1 in Braido ([21]) implies that $-d(y^t) < \frac{R}{1-\rho} \equiv k_{2,min}$ uniformly in $y^t \in \Omega$ with ρ sufficiently close to 1. From assumption 1-a3), 1-b), 1-c) and 1-d), equation (67) implies $p(y^t) = \frac{A_2(c_2(y^t))}{A_1(c_1(y^t))}$. Definition 1-2), equation (67), assumption 1-e), 1-f) then imply $p(y^t) \in [\frac{y^N}{cu_1}, \frac{y^N}{cl_1}] \equiv K_3 \equiv [k_{3,min}, k_{3,max}] \quad y^t - a.e..$ Then, the collateral constraint implies $d(y^t) \leq \kappa(y_{max} + k_{3,max}y^N) \equiv k_{2,max}$. Then, $d(y^t) \in [k_{2,min}, k_{2,max}] \equiv K_2 \quad y^t - a.e..$ Finally, using K_2 and K_3, K_1 can be derived using equation (2) and Definition 1-2). Using these results, it is straightforward to verify that $lim_{t \longrightarrow \infty} \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$. Note that the integral in E_t is taken with respect to either the density associated with $\chi(y_t)$ or the $y_t jth$-row in the transition matrix in case Y has finite cardinality. Thus, this result also holds $y^t - a.e..$ Then, the arguments in Lemma 2 in Kubler and Schmedders ([41]) hold, which implies that equations (67) and (68) together with the terminal condition on $\beta^t E_t(U'(A(c_{t+1}))A_1(c_1(t+1)))$ are necessary to definition 1, as desired. ∎

Proof of Theorem 4

Proof. As in Kubler and Schmedders ([41]), we will start with a truncated economy $t = 0, ..., T$ and then extend the argument by induction. In order to show the theorem, we need to rewrite the conditions in definition 1. Any SCE satisfies conditions A and B: Condition A

$$Max_{d(y^t)} \sum_t \sum_{y^t} U(c_1(y^t), y^N)\mu(y^t)$$

$$s.t$$

$$c_1(y^t) = y_t^T - d_t + min\left\{d_{t+1}/R, R^{(-1)}\kappa(y_t^T + p(y^t)y^N)\right\}$$

and Condition B

$$p(y^t) = \frac{A_2(y^N)}{A_1(c_1(y^t))}$$

Because of the linearity of the restriction, we can substitute c_1 into the objective function in condition A. Thus, because of lemma 3, the maximization problem is only restricted by the fact that $d(y^t) \in K_2$ for all y^t. Thus, $\{d_{t+1}(y^t)\}_{y^t} \in K_2^{(Y)^T}$, where $(Y)^T$ is the number of possible nodes. Note that given p, this is a strongly concave problem restricted by a continuous correspondence. By Berge's maximum theorem, we know that $d_{t+1}(d_0, y^t, p)$ will be a continuous function of $p \in K_3^{(Y)^T}$ for all $d_0 \in K_2$ and $y^t \in (Y)^T$. Then we can use Condition B to define a continuous operator for $p((y)^T)$, P, from a compact sect $K_3^{(y)^T}$ to itself, which has a fixed point. To see this note that in the unconstrained case, the system of equations generated by conditions A and B are block recursive (i.e., we first solve for d_{t+1} give d_t then compute p_t for $t = 0, 1, ..T$). Thus, the proof in immediate. When the collateral constrains binds, condition B generates a finite number of roots $p_*(y^t, R_n)$, where R_n stands for the n-th root (this is extensively discussed in section 5). That is, $p_*(y^t, R_n) = P(p_*(y^t, R_n))$. Due to the compactness of equilibrium, it is possible to locally generate an arbitrary large sequence of prices $p_j(y^t, R_n)$ which iteratively (i.e., $p_{j+1}(y^t, R_n) = P(p_j(y^t, R_n)))$ converges to $p_*(y^t, R_n)$ for any n. This is possible as locally a continuous function is either increasing or decreasing.

To show sufficiency of equations (67), (68) and the transversality condition, note that Condition A defines a strongly concave problem with a unique solution. Thus, the set of solutions to Condition A given $p(y^T)$ using Berge's theorem is equivalent to the solutions found using (67), (68) and the transversality condition. To see this, let us define the correspondence $\Gamma(y_t^T, d_t) = \left\{d_{t+1} \in K_2 : K_{LB,2} \leq d_{t+1} \leq R^{-1}\kappa(y_t^T + p_t y^N)\right\}$, where $K_2 = [K_{LB,2}, K_{UB,2}]$, and the return function $U(c_1(y^t), y^N) \equiv F(y_t^T - d_t + R^{-1}d_{t+1})$. Then, condition A can be seen as maximization problem with a control given by the sequence of iteratively feasible debt: $\left\{d_{t+1}(y_0^T, d_0)\right\}$ $d_{t+1} \in \Gamma(y_t^T, d_t)$ for a given sequence of prices $\{p_t\}$. This problem can be characterized using the sufficient conditions in Kamihigashi ([37]): F is continuous in $K_2 \times K_2$ for any y_t^T, given lemma 3 is bounded, $F_{-d_t} \geq 0$ and it is strictly concave. To see this last property note that, even tough the Hessian of F in $K_2 \times K_2$ is zero, the elements in the diagonal are both negative. Thus, this function is concave. However, for any given d_t, there is a unique d_{t+1} for each c_1 and U is strictly concave in c_1. Thus, (67), (68) and the transversality condition are sufficient for a maximal $\left\{d_{t+1}(y_0^T, d_0)\right\}$. Then, the arguments above imply that adding condition B suffice to show the existence of a SCE. ■

Proofs for Section 3

Proof of Lemma 7

Proof. The proof of part takes place in five steps. We first show the operator $A(c; C^T(S))(d, y)$ as for each $C^T(S) \in \mathbf{C}^f$, $A(c; C^T(S))(d, y) : \mathbf{C}^p \rightarrow \mathbf{C}^{p*}$, is well-defined, where $\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; C^T) \subset \mathbf{C}^p$, and given by:

$$\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; C^T) = \{c \in \mathbf{C}^p |\ c = \inf\{\hat{c}(d, y), C_c^T(D, Y, C_i^T(S)), \hat{c} \in \mathbf{C}^p, C^T(S) \in \mathbf{C}^f\} \quad (69)$$

is closed $\mathbf{C}^p$. Second, we show the mapping $A(c, C^T(S))(d, y)$ is jointly monotone on $\mathbf{C}^p \times \mathbf{C}^f$, and order continuous in $c \in \mathbf{C}^p$ for each $C^T \in \mathbf{C}^f$. Third, we show the greatest fixed point of $A(c; C^T(S))(d, y)$ (denoted for now by $c^*(C^T(S))(d, y)$) is strictly positive, can be computed by successive approximations from an initial $c_0 = c_{\max}$ for each $C^T \in \mathbf{C}^f$. The fourth step, we show the greatest fixed point is increasing in $C^T(S)$ on $\mathbf{C}^f$. Finally, in the fifth step, we show $c^*(C^T(S), \beta, R, \kappa)(d, y)$ is the *unique* strictly positive fixed point in $\mathbf{C}^p$ of $A(c; C^T(S))(d, y)$ for each $C^T \in C^f$.[50]

Step 1: $A(c; C^T(S))(d, y) : \mathbf{C}^p \to \mathbf{C}^{p*}$. Fix $C^T \in \mathbf{C}^f$, $c \in \mathbf{C}^p$, and $s = (d, y, S)$. First, the operator $A(c; C^T(S))(d, y)$ is well-defined. So see this, observe when $c(d, y) \in \mathbf{C}^p$, $c(d, y) = 0$ for any state (d, y), $C(d', y'; c, C^T) = 0$, we define $x^*_{uc}(s^e, c, C^T) = A(c; C^T(S))(d, y) = 0$. So consider the case when $c \in \mathbf{C}^p$, $c(s^e) > 0$. As $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$, the mapping $Z^*_{uc}(x, s^e; c, C^T)$ in equation (37) is strictly decreasing and continuous in x, for any $(d, y, S; c, C^T)$. Compute an implicit mapping $x^*_{uc}(d, y, S; c, C^T)$ in the following equation:

$$Z^*_{uc}(x^*_{uc}(d, y, S; c, C^T), s; c, C^T) = 0$$

If $x^*_{uc}(d, y, S; c, C^T)$ exists, as Z^*_{uc} is strictly decreasing and continuous in x under Assumption 1, $x^*_{uc}(d, y, S; c, C^T)$ will necessarily be unique (hence, a function). When $x \to 0$, $Z^*_{uc}(x, s^e, c, C^T) \to \infty$ by the Inada condition in Assumption 2. Further, as x gets sufficiently large, $C((R(x - y^T + d), y', R(x - y^T + d, y') \to 0$, hence $Z^*_{uc} \to -\infty$. Then, by the intermediate value theorem, $x^*_{uc}(d, y, S, c, C^T)$ exists (hence, it is well-defined as a function).

Next, we show $x^*_{uc}(d, y, S; c, C^T) \in \mathbf{C}^{p*} \subset \mathbf{C}^p$. We first show $A_{uc}(c; C^T(S))(d, y) = x^*_{uc}(d, y, S; c, C^T) \in \mathbf{C}^p$. Again, when $c(d, y) \in \mathbf{C}^p$, $c(d, y) = 0$ in any state (d, y), $\Rightarrow C(c, C^T) = 0$; hence, $x^*_{uc}(d, y, S; c, C^T) = 0 \in \mathbf{C}^p$. Therefore, consider the case when $c \in \mathbf{C}^p$, $c(s^e) > 0$. As $C^T \in C^f$, for fixed $c \in \mathbf{C}^p$, Z^*_{uc} in (37) is (strictly) decreasing in d, (strictly) increasing in y, and strictly decreasing in x; hence, at such $s = (d, y, S)$, the root $x^*_{uc}(d, y, S; c, C^T)$ is decreasing in d, and increasing in y. Further, when $d_2 \geq d_1$ and $y_1 \geq y_2$, by the concavity of utility in Assumption 1, we have from the definition of the $x^*_{uc}(d, y, S, c, C^T)$ in Z^*_{uc} the following inequality

$$\frac{U_1(x^*_{uc}(d_1, y_1, S; c, C^T), y^N)}{R} \leq \int \beta U_1(C(R(x^*_{uc}(d_2, y_2, S; c, C^T) - y_2^T + d_2), y', R(x^*_{uc}(d_2, y_2, S; c, C^T) - y_2^T + d_2), y')\chi(dy')$$

hence, for the root $x^*_{uc}(d, y, S; c, C^T)$ must make the right side of the above expression fall at $x^*_{uc}(d_2, y_2, S; c, C^T)$ in a new solution, which implies:

$$x^*_{uc}(d_1, y_1, S; c, C^{T*}) - y_1^T + d_1 \geq x^*_{uc}(d_2, y_2, S; c, C^{T*}) - y_2^T + d_2$$

or

$$y_1^T - d_1 - x^*_{uc}(d_1, y_1, S; c, C^{T*}) \leq y_2^T - d_2 - x^*_{uc}(d_2, y_2, S; c, C^T)$$

Therefore, for each $C^T \in C^*$, $A_{uc}(c; C^T(S))(d, y) = x^*_{uc}(d, y, S; c, C^T) \in C^p$.

Finally, as $A_{uc}(c; C^T)(s^e) \in C^p$, and $A_c(C^T)(S)$ is independent of (d, y) at $C^T(S) \in C^f$, $A(c; C^T)(s^e) = \inf\{A_{uc}(c; C^T)(s^e), A_c(C^T)(S)\} \in \mathbf{C}^{p*}$.Therefore, we conclude $A(c; C^T(S))(d, y) : \mathbf{C}^p \to \mathbf{C}^{p*}$.

Step 2: $A(c, C^T(S))(d, y)$ *is monotone (increasing) on* $C^p \times C^f$. Take $x_1 = (c_1, C_1^T)$ and $x_2 = (c_2, C_2^T) \in \mathbf{C}^p \times \mathbf{C}^f$, with $x_1 \leq x_2$ under the pointwise partial order on the product space $\mathbf{C}^p \times \mathbf{C}^f$. First, consider the case $0 \leq x_1 \leq x_2$, where in some state (d, y, S), either $0 = c_1(d, y)$ or $0 = C_1^T(S)$, Then, by definition of the operator $A(c, C^T(S))(d, y)$, $A(c_1, C_1^T)(d, y, S) = 0 \leq A(c_2, C_2^T)(d, y, S)$. So, now consider the case where $0 < x_1(d, y) \leq x_2(d, y)$, so in all states, $0 < c_1(d, y)$ and $0 < C_1^T(S)$. Then, we have from the

[50] This last step will also imply that the unique strictly positive fixed point is continous in the topology of pointwise convergence in C^T. As it will also be isotone on C^T, the resulting second step operator will be *order continuous* on C^T. That mean in main theorem of this section, *all* our arguments can be make constructive as mentioned in section 3. See Pierri and Reffett ([64]) for a discussion.

definition of x^*_{uc} in Z^*_{uc} the following inequality:

$$\frac{U_1(x^*_{uc}(d,y,S;c_1,C_1^T),y^N)}{R} =$$
$$\int \beta U_1(C_1(R(x^*_{uc}(d,y,S;c_1,C_1^T)-y^T+d),y',R(x^*_{uc}(d,y,S;c_1,C_1^T)-y_2^T+d_2),y')\chi(dy')$$
$$\geq \int \beta U_1(C_2(R(x^*_{uc}(d,y,S;c_1,C_1^T)-y^T+d),y',R(x^*_{uc}(d,y,S;c_1,C_1^T)-y_2^T+d_2),y')\chi(dy')$$

where for $i=1,2$, the subscript on continuation consumption is used to denote.

$$C_i(c,C^T)(d,y,S) = \inf\{c_i(d,y), C_c^T(D,Y,C_i^T(S))\}$$

where recall $C_c^T = (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(D,Y)Y^N$. Therefore, as Z_e^* is strictly falling in x, we have

$$x^*_{uc}(d,y,S;c_1,C_1^T) \leq x^*_{uc}(d,y,S;c_2,C_2^T)$$

Then, if the implied debt at $d_{x_{uc}}(d,y,S;c_1,C_1^T) \leq \kappa(Y^T - D + p(C_1^T)Y^N$, then

$$\begin{aligned} A(c_1,C_1^T(S))(d,y) &= A_{uc}(c_1,C_1^T(S))(d,y) \\ &= x^*_{uc}(d,y,S;c_1,C_1^T) \\ &\leq x^*_{uc}(d,y,S;c_2,C_2^T) \end{aligned}$$

else,

$$\begin{aligned} A(c_1,C_1^T(S))(d,y) &= (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C_1^T(S))Y^N \\ &\leq (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C_2^T(S))Y^N \end{aligned}$$

In either case, for $A(c_2,C_2^T(S))(d,y)$, we have $A(c_1,C_1^T(S))(d,y) \leq A(c_2,C_2^T(S))(d,y)$. So the operator $A(c,C^T(S))(d,y)$ is monotone.

Next, we prove $A(c;C^T(S))(d,y) : \mathbf{C}^p \to \mathbf{C}^{p*}$ is order continuous for each fixed $C^T(S) \in \mathbf{C}^f$. First, some definitions. Let X be a countably chain complete partially ordered set[51] and $X_c = (x_n)_{n\in\mathbf{N}} \subset X$, $x_n \in X$, be a countable chain. We say a operator $A : X \to X$ for is *order continuous* if for any countable chain $X_c \subset X$, $A(x)$ (a) sup-preserving: $A(\vee X_c) = \vee A(X_c)$ and (b) inf-preserving: $A(\wedge X_c) = \wedge A(X_c)$. We remark, order continuous operators are necessarily isotone (e.g., Dugundji and Granas ([32], p. 15)). We now show for each $C^T(S) \in \mathbf{C}^f$, $A(c;C^T(S))(d,y)$ preserves sup operations; a similar argument works for preserving inf operations. Fix the state (d,y), and $C^T(S) \in \mathbf{C}^f$, and denote by $C_c = (c_n(d,y))_{n\in\mathbf{N}}$, $c_n(d,y) \in \mathbf{C}^p$ any countable chain in $\mathbf{C}^p$. Define $\vee C_c(d,y) \in \mathbf{C}^p$ and $\vee A(C_c;C^T(S))(d,y) \in \mathbf{C}^{p*}$, which both exist in $\mathbf{C}^p$ (resp, $\mathbf{C}^{p*}$) are both complete lattices (hence, countably chain complete). If in any state (d,y), $\vee C_c(d,y,S) = 0$, then $\vee A(C_c;C^T(S))(d,y) = A(\vee C_c;C^T(S)) = 0$. Therefore, assume for every state (d,y,S), $\vee C_c(d,y) > 0$. Then, we have the following inequalities for continuation tradeables consumption $C(c_n;C^T) = \inf\{c_n(d,y), C_c^T(D,Y,C^T(S))\}$

$$\begin{aligned} C(\vee C_c) &= C(\vee c^n;C^T) \\ &= \inf\{\vee c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N\}) \\ &= \vee \inf_n(\{c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))Y^N\}) \\ &= \vee C(c^n;C^T) = \vee C(C_c;C^T(S)) \end{aligned}$$

where in the second line $\vee c_n(d,y)$ is computed, and then the infimum over two continuous functions $(\vee c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N)$ is taken over a compact set $(d,y,S) \in \mathbf{D}\times\mathbf{Y}\times\mathbf{S}$, and

[51] Let X be a partially ordered set. We say X is countably chain complete if for all countable subset X_c that are a chain (i.e., for no two elements $x_1, x_2 \in X_c$, x_1 and x_2 are ordered), $\vee X_c \in X$ and $\wedge X_c \in X$.

hence continuous by Berge's theorem, in the third line, $\inf_n$ is computed pointwise over $(d, y) \in D \times Y$ (a compact set and hence continuous) at each $n \in \mathbf{N}$, and this collection is then increasing pointwise in n as C_c is a countable chain) and the sup is taken over $n \in \mathbf{N}$. Then, the remaining equalities follow from p continuous, and the fact that sup and inf operations over two continuous functions are each continuous over the compact set $(d, y, S) \in \mathbf{D} \times \mathbf{Y} \times \mathbf{S}$ by Berge's maximum theorem.
Using these facts, and substituting into the definition of $Z^*_{uc}(x, d, y, S; c, C^T)$, we have for the root $x^*_{uc}(d, y, S, c, C^T)$ the following equalities:

$$\begin{aligned} Z^*_{uc}(x^*_{uc}(d, y, S; \vee c_n, C^T), d, y, S; \vee c_n, C^T) &= \vee Z^*_{uc}(x^*_{uc}(d, y, S; \vee c_n, C^T), d, y, S; c_n, C^T) \\ &= \vee Z^*_{uc}(x^*_{uc}(d, y, S; c_n, C^T), d, y, S; c_n, C^T) \\ &= Z^*_{uc}(\vee x^*_{uc}(d, y, S; c_n, C^T), d, y, S; c_n, C^T) \end{aligned}$$

where the first equality follows from $U_1(c, y^N)$ continuous and $C(\vee c_n, C^T) = \vee C(c_n, C^T)$, the second line follows from Z^*_{uc} continuous (pointwise) in (x, c_n) for fixed C^T, the third line follows from Z^*_{uc} continuous in x. Then, noting that for any state where collateral constraints do not bind, we have $x^*_{uc}(d, y, S, c_n, C^T) \leq A_c(C^T)(S)$, our operator $A(c, C^T(S))(d, y)$ is for each $n \in \mathbf{N}$ defined as:

$$A(c_n, C^T(S))(d, y) = \inf_n \{x^*_{uc}(d, y, S; c_n, C^T), A_c(C^T)(S)\}$$

so we have the following:

$$\begin{aligned} A(\vee c_n; C^T(S))(d, y) &= \inf\{x^*_{uc}(d, y, S; \vee c_n, C^T), A_c(C^T)(S)\} \\ &= \inf\{\vee x^*_{uc}(d, y, S; c_n, C^T), A_c(C^T)(S)\} \\ &= \vee \inf_n \{x^*_{uc}(d, y, S; c_n, C^T), A_c(C^T)(S)\} \\ &= \vee A(c^n, C^T(S))(d, y) \\ &= \vee A(c_n; C^T(S))(d, y) \end{aligned}$$

where the second equality follows again from $U_1(c, y^N)$ is continuous and $C(\vee c_n; C^T) = \vee C(c_n; C^T)$ for each C^T,and for the third equality again uses the fact that $\inf_n$ here is an increasing pointwise in n, and the sup is then taken over $n \in \mathbf{N}$. Hence, $A(c; C^T(S))(d, y)$ is order continuous in $\mathbf{C}^p$ for each fixed $C^T \in \mathbf{C}^f$, which completes the proof of Step 2.

Remark: Before proceeding to step 3, we mention that as equilibrium fixed point comparative statics be an important question in Steps 4 and 5, for the remaining steps of the proof of this lemma, we add to the notation for our operator for the parameters of interest, and remark that the operator $A(c; C^T(S), \beta, R, \kappa)(d, y)$ is increasing in κ, and decreasing in (β, R) for fixed (c, C^T, d, y, S). To see this, noting $c \in \mathbf{C}^p$ is decreasing in d, U_1 is decreasing in c under assumption 1, Z^*_{uc} in (37) is decreasing in (R, β). So, the root $x^*_{uc}(d, y, S, c, C^T; \beta, R, \kappa)$ is decreasing in (β, R). Further, $A_c(C^T; R, \kappa)$ is decreasing in R. As our operator is defined as the infimum of two decreasing mappings, $A(c; C^T(S), \beta, R, \kappa)(d, y)$ is decreasing in (β, R). As Z^*_{uc} is independent of κ, but $A_c(C^T; R, \kappa)$ is increasing in κ, $A(c; C^T(S), \beta, R, \kappa)(d, y)$ is increasing in κ.

Step 3. *Existence and computation he greatest fixed point of* $A(c; C^T(S), \beta, R, \kappa)(d, y) \in \mathbf{C}^{p*} \subset \mathbf{C}^p$. Fix $C^T \in \mathbf{C}^f$, and denote by $\Psi_A(C^T(S), \beta, R, \kappa)(d, y) \subset \mathbf{C}^{p*}$ the set of fixed points of mapping of $A(c; C^T(S), \beta, R, \kappa)(d, y) \in \mathbf{C}^{p*} \subset \mathbf{C}^p$.[52] By definition, the least fixed point is trivial, and is $c^* = 0$ for all $y \in Y$. By step 2 above, $A(c; C^T(S), \beta, R, \kappa)(d, y)$ is order continuous on $\mathbf{C}^p$. Further, $A(c^{\max}; C^T(S))(d, y) \leq c_{\max}$ (with strict inequality for some states (d, y)). Hence, by the Tarski-Kantorovich theorem (e.g, Dugundji and Granas ([32], p.15), the greatest fixed point $c^*(C^T(S))(d, y)$ can be computed as:

$$\wedge A^n(c^{\max}; C^T(S), \beta, R, \kappa)(d, y) \to c^*(C^T(S), \beta, R, \kappa)(d, y) > 0$$

[52] Note, as as $\mathbf{C}^{p*}$ as $\mathbf{C}^{p*}$ is a subcomplete sublattce in $\mathbf{C}^p$, $\mathbf{C}^p$ is a complete lattice, this implies the fixed point of the mapping $A(c; C^T(S), \beta, R, \kappa)(d, y) \in \mathbf{C}^{p*}$.

where the strict positivity of $c^*(C^T(S),\beta,R,\kappa)(d,y)>0$ follows from the Inada condition on $U_1(c;y^N)$ in its first argument, and we note the dependence of $c^*(C^T(S),\beta,R,\kappa)(d,y)$ on deep parameters for later reference. That proves the existence of a strictly positive greatest fixed point.

Step 4. *Fixed point comparative statics.* By standard fixed point statics argument for order continuous operators, the greatest fixed point $c^*(C^T(S),\beta,R,\kappa)(d,y)$ is increasing in $(C^T(S),\kappa)$, and decreasing in (β,R). Then, the associated stationary policy function for tradeable consumption conditioned on the equilibrium collateral constraint be fixed at $C^T(S)\in\mathbf{C}^f$ is given by

$$C(C^T(S),\beta,R,\kappa)(d,y)=\inf\{c^*(C^T(S),\beta,R,\kappa)(d,y),(1+\frac{\kappa}{R})Y^T-D+\frac{\kappa}{R}p(C^T(S))y^N\}$$

which is continuous in $(d,y)\in\mathbf{D}\times\mathbf{Y}$ by Berge's theorem, increasing in $(C^T(S),\kappa)$ and decreasing in (β,R). This completes to proof of the fixed point comparative statics claim in the lemma.

Step 5: $c^*(C^T(S),\beta,R,\kappa)(d,y)$ *the unique strictly positive fixed point.* This follows from an application of Corollary 4.1 in Li and Stachurski ([43]) for each $C^T\in\mathbf{C}^f$. To see this, for fixed $C^T\in\mathbf{C}^f$ put

$$\begin{aligned}\varsigma(d,y)&=u'((1+\frac{\kappa}{R})Y^T-D+\frac{\kappa}{R}p(C^T(d,y))y^N) \qquad (70)\\&=A_c(C^T)(d,y)\end{aligned}$$

and restrict the first step operator $A(c;C^T(S),\beta,R,\kappa)(d,y)$ to the set $\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)\subset\mathbf{C}^p_{++}=\{c\in\mathbf{C}^p|c(d,y)>0\}$. where

$$\mathbf{C}^{p*}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)=\{c|\ c=\inf\{\hat{c}(d,y),\varsigma(d,y)),\hat{c}\in\mathbf{C}^p_{++}\}$$

where in our notation we make explicit the dependence of the the space $\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$ on the upper bound in (70). Let c_1 and c_2 be elements of $\mathbf{C}^{p*}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$. Equipped the space the $\mathbf{C}^{p}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma(d,y))$ with the norm

$$\rho(c_1,c_2)=\parallel u'_\varsigma\circ c_1-u'\circ c_2\parallel$$

where $\parallel u'\circ c_1-u'\circ c_2\parallel<\infty$, where $u'(c)=U'(A(c))A_1(c^T,y^N)$ is strictly decreasing in c^T under Assumption 1, and give $\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e)\subset\mathbf{C}^{p}_{++}(\mathbf{D}^e\times\mathbf{Y}^e)$ its relative distance structure.

First, from the arguments in Step 1 of this proof, $A(c;C^T(S),\beta,R,\kappa)(d,y)$ maps $\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$ into $\mathbf{C}^{p*}_{+}(\mathbf{D}^e\times\mathbf{Y}^e)$. Also, when $c(d,y)\in\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$ is strictly positive, $A(c;C^T(S),\beta,R,\kappa)(d,y)>0$ (by the Inada condition in Assumption 2). By Li and Stachurski ([43], Proposition 4.1.a), the pair ($\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$,$\rho$) is a complete metric space. As $\beta R<1$, by Li and Stachurski ([43], Proposition 4.1.c), for each $A(c;C^T(S),\beta,R,\kappa)(d,y)$ is a contraction of modulus $0<\beta R<1$ in ($\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$,$\rho$). Then, by the contraction mapping theorem, $A(c;C^T(S),\beta,R,\kappa)(d,y)$ has exactly one fixed point in ($\mathbf{C}^{p*}_{++}(\mathbf{D}^e\times\mathbf{Y}^e;\varsigma)$,$\rho$). So, $c^*(C^T(S),\beta,R,\kappa)(d,y)$ is the unique strictly positive fixed point of $A(c;C^T(S),\beta,R,\kappa)(d,y)$ for $C^T\in\mathbf{C}^f$. ■

Corollary 16 *The mapping $A^*(C^T;\beta,\kappa,R)(s^e))$ in (46) is order continuous on $\mathbf{C}^f$.*

Proof. As the operator in step 5 $A(c;C^T(S),\beta,R,\kappa)(d,y)$ is easily shown to be continuous in $C^T(S)\in\mathbf{C}^f$ in the topology of pointwise convergence, by the Bonsall-Nadler theorem on parameterized contractions, $c^*(C^T(S),\beta,R,\kappa)(d,y)$. (e.g., see Nadler ([57], Theorem 2 and Lemma, p. 581)). As $c^*(C^T(S),\beta,R,\kappa)(d,y)$ is also monotone increasing (in pointwise partial orders) by Step 4, $c^*(C^T(S),\beta,R,\kappa)(d,y)$ is order continuous on $\mathbf{C}^f$.

■

Proof of Lemma 6

Proof. As $c(d,y)$ is decreasing (resp., increasing) in d (resp., y) such that $-d'(d,y)=R(y^T-d)-c(d,y)$ is decreasing (resp., increasing) in d (resp., in y), we have $|c(d',y')-c(d,y)|\leq R\ |(y'-d')-(y-d)|$ when $(d',y')\geq(d,y)$, hence $\mathbf{C}^p(\mathbf{S}^e)$ is an (uniformly) equicontinuous collection of continuous functions. It is therefore a compact set in the topology of uniform convergence (and hence chain complete (e.g., Amann ([5], lemma 3.1)). Further, let $\mathbf{C}_1=\{(c_{i\in I}(d,y)\}\subset\mathbf{C}^p(\mathbf{S}^e)$ be an arbitrary collection. Then the monotonicity properties of $\{(c_{i\in I}(d,y)\}$ and its associate $\{(-d_{i\in I}(d,y)\}$ are preserved under pointwise sup and infs. Let $\vee\mathbf{C}_1 = \vee c(d,y)$ and $\wedge\mathbf{C}_1 = \wedge c(d,y)$.Hence, $|\vee c(d',y') - \vee c(d,y)| \leq R\,|(y'-d')-(y-d)|$ and $|\wedge c(d',y') - \wedge c(d,y)| \leq R\,|(y'-d')-(y-d)|$. So $\vee\mathbf{C}_1$ and $\wedge\mathbf{C}_1$for any collection $\mathbf{C}^p(\mathbf{S}^e)$. So $\mathbf{C}^p(\mathbf{S}^e)$ is a complete lattice. ■

Proof of Lemma 8

Proof. Fixing (D,Y), the collection $c^*(d,y;C^T(D,Y)) \in \mathbf{C}^{p*}$ forms an equicontinuous collection by Lemma 6. Also, fixing (d,y), as $C^T \in \mathbf{C}^f$, $C^T \in [C_m(S), c_{\max}]$, $C^T(D,Y)$ is decreasing (resp, increasing) in D (resp, Y) such that the associated debt $D'(D,Y) = \kappa(Y^T + p(C^{T*}(D,Y))Y^N)$, $D'(D,Y)$ is also decreasing (resp., increasing) in D (resp., Y). Notng that $p(C)$ is continuous under assumption 1, this implies for fixed (d,y), $c^*(d,y;C(D,Y))$ forms on equicontinuous collection (noting that $p(C)$ is locally Lipschitzian and uniformly continuous in C under assumption 1). So, let $\mathbf{C}_1 = (c^*_{i\in I}(d,y,C(D,Y)) \subset \mathbf{C}^*$ be any arbitrary subset in $\mathbf{C}^*$. When $d = D, y = Y$, as $(c^*_i(d,y,C(d,y))$ is an equicontinuous functions. Further,, the monotonicity properties over (d,y,D,Y) where $d = D$, $y = Y$ pointwise for each $c^*_{i\in I}(d,y,C(D,Y)$ and associated $\{D'_{i\in I}(d,y,D,Y) = \kappa(Y^T + p(C(d,y,D,Y))Y^N$ are preserved for $\vee\mathbf{C}_1(d,y,D,Y)$ and $\wedge\mathbf{C}_1(d,y,D,Y)$. Therefore, $\vee\mathbf{C}_1 \in \mathbf{C}^*$ and $\wedge\mathbf{C}_1 \in \mathbf{C}^*$. Therefore, $\mathbf{C}^*$ is a complete lattice. ■

Proof of Theorem 9.

Proof. (i) Let $\Psi^*(R,\kappa,\beta) \subset \mathbf{C}^*$ be the set of fixed points of the mapping $A^*(C^T;\beta,R,\kappa)(s^e)) = \inf\{c^*(C^T(S);\beta,R,\kappa)(d,y),\ A_c(C^T)(s^e)\}$ defined in (46). That $A^*(C^T,\beta,R,\kappa)(s^e)) \in \mathbf{C}^*$ is immediate, as (a) by construction for fixed S, when $d = D$, $y = Y$, $c^*(C^T(S);\beta,R,\kappa)(d,y) \in \mathbf{C}^p$, and (b) when (d,y) is fixed, as $c^*(C^T(S);\beta,R,\kappa)(d,y)$ is increasing in $C^T \in \mathbf{C}^*$, and C^T is increasing in Y, and decreasing in D, $\inf\{c^*(C^T(S);\beta,R,\kappa)(d,y),\ A_c(C^T)(S))\}$ is increasing in Y, and decreasing in D. Further, if s^e is a collateral constrained state, then $d'_{A^*(C^T)}(s^e) = \kappa\{y^T + p(C^T(d,y))y^N)$ is increasing in y, and decreasing in d. Finally, as $C(D,Y) \in \mathbf{C}^f$, $A_c(C^T)(s^e) = (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N \geq 0$, so $A_c(C^T)(s^e) \in \mathbf{C}^f$. Therefore, we have $A^*(C^T,\beta,R,\kappa)(s^e)) \in \mathbf{C}^*$. Then, by lemma 7, step 4, as operator $A^*(C^T)(s^e))$ is monotone increasing on $\mathbf{C}^*$, and by Lemma 8, $\mathbf{C}^*$ is a nonempty complete lattice. Then, by Tarski's theorem ([79], theorem 1), $\Psi^*(R,\kappa,\beta)$ is a nonempty complete lattice. As all the fixed points are strictly positive, each fixed point induces a RCE.
(ii) Noting its dependence on deep parameters now, as $A^*(C^T,\beta,R,\kappa)(s^e))$ is decreasing in (β,R), and increasing in κ, by Veinott's parameterized version of Tarski's theorem ([81], chapter 4; also, [80], Theorem 2.5.2) , the least and greatest selections of $\Psi^*(R,\kappa,\beta)$ exist as fixed points, and are decreasing in (β,R), and increasing in κ.
(iii) $A^*(C^T,\beta,R,\kappa)(s^e)) = \inf\{c^*(C^T(S);\beta,R,\kappa)(d,y),\ A_c(C^T)(s^e)\}$ is order continuous under pointwise partial orders on $\mathbf{C}^f$ as (a) $A_c(C^T)(s^e) = 1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(d,y))y^N$ is pointwise continuous and monotone ($p(C)$ is continuous and monotone increasing under Assumption 1), (b) by Corollary 16 $c^*(C^T(S),\beta,R,\kappa)(d,y)$ is order continuous, and hence, (c) $\inf\{c^*(C^T(S);\beta,R,\kappa)(d,y),\ A_c(C^T)(s^e)\}$ pointwise continuous and increasing by Berge's theorem (as the state space is compact).
Next, we construct a strictly positive tradeables consumption function $C_m \in \mathbf{C}^*$ that is a lower subsolution needed to compute and compare the the least fixed point (i.e., we construct a function C_m that maps up under A^* (i.e, has $0 < C_m \leq A^*(C_m;\beta,\kappa,R)(s^e))$, $C_m \in \mathbf{C}^*$). We do this by showing that for each β,κ,R, the iterations $\inf_n A^{*n}(C_0)(s^e) \to C_m(s^e) \in \mathbf{C}^*$ when starting the iterations from $C_0 = 0$ and for any initial s^e such that $\wedge y - d \geq 0$. That is, let $C^T(S) = 0$ in all states. Then, for any state $s = (d,y,D,Y)$ when $d = D$, $y = Y$, and for any (y^T,d) such $\wedge y^T - d \geq 0$, noting that tomorrow's "unconstrained tradeables consumption" is $c(d',y') = c^*(0)(s^e)$ (which is the unique strictly positive fixed point in lemma 7):

$$Z^*_{uc}(x^*_{uc}(0)(s^e),s^e;c,C^T), = \frac{U_1(x^*_{uc}(0)(s^e),y^N)}{R} - \\ \beta\int U_1(c^*(0)(R(x^*_{uc}(0)(s^e) - y^T + d),y'),(1+\frac{\kappa}{R})y^{T'} - R(x^*_{uc}(0)(s^e) - y^T - d))\chi(dy')$$

where by the Inada condition, $x^*_{uc}(0)(s^e) > 0$, and compute the implied debt $d'(0)(s^e)$ in all states. If in this state s^e, $d'(0)(s^e) \leq \kappa y^T$, $A^*(0)(s^e) = x^*_{uc}(0)(s^e)$, else

$$A_c(0)(s^e) = (1 + \frac{\kappa}{R})y^T - d$$

Notice, $C_0 = 0 < A^*(0)(s^e)$ in all *unconstrained* states s^e for all states that have $y^T - d \geq \wedge y^T - d \geq 0$. To continue the iterations, let $d_{\max,1}$ be the value of $d'(0)(s^e)$ when $y^T = \wedge y^T$ such that $d'(0)(s^e) = \kappa \vee y^T$, and now let the collateral constraint for the next iteration be

$$d' \leq (1 + \frac{\kappa}{R})y^{T'} + p(A^*(0)(s^e))y^{NT}$$

where $\kappa y^{T'} < \kappa(y^T - p(A^*(0)(s^e))y^{Nt})$ (i.e., the collateral constraint at $A^*(0)(s^e) = C^T_1(s^e)$ will allow for strictly more uncollateralized debt than at $C^T = 0$. Now, recursively, when $\wedge y^T - d \geq \wedge y^T - d_{\max,n}$, define $A^*(C^T_n)(s^e)$ where $C^T_n = A^*(C^T_{n-1})$ in a similar manner: i.e, compute

$$Z^*_{uc}(x^*_{uc}(C^T_{n-1})(s^e), s^e; c, C^T_{n-1}), = \frac{U_1(x^*_{uc}(C^T_{n-1})(s^e), y^N)}{R} - \beta \int U_1(c^*(C^T_{n-1})(R(x^*_{uc}(C^T_{n-1})(s^e) - y^T + d), y'), (1 + \frac{\kappa}{R})y^{T'} - C^T_{n-1}(R(x^*_{uc}(C^T_{n-1})(s^e) - y$$

where now we have $x^*_{uc}(C^T_{n-1})(s^e) > x^*_{uc}(C^T_{n-2})(s^e)$ in all states $y^T - d \geq \wedge y^T - d_{\max,n-1}$, and compute the implied debt $d'(C^T_{n-1})(s^e)$ for $^*_{uc}(C^T_{n-1})(s^e)$ in each state. If $d'(C^T_{n-1})(s^e) \leq \kappa(y^T + p(C^T_{n-1}(s^e))y^{NT}$, $A^*(C^T_{n-1})(s^e) = x^*_{uc}(C^T_{n-1})(s^e)$, else

$$A_c(C^T_n)(S) = (1 + \frac{\kappa}{R})y^T - d + p(C^T_{n-1}(s^e))y^{NT}$$

notice $0 \leq A^*(C^T_0)(s^e) \leq ... \leq A(C^T_{n-1}(s^e))$, and in *unconstrained* states s^e we have $0 < A^*(0)(s^e) < ... < A^*(C^T_{n-1}(s^e))$.
To complete the construction, compute $\inf_n A^{*n}(0)(s^e) \to C_m(s^e)$ which exists are $A^*(C^T)$ is isotone (order continuous, actually), and let $d_{\max} = \sup_n d_{\max,n}$. Further, by construction $C_m(s^e)$ is strictly positive in all states and satisfies all the conditions in $\mathbf{C}^*$.
We then are now ready to complete the proof of the theorem. By the definition of $C_m(S)$, $0 < C_m(S) \leq A^*(C_m; \beta, \kappa, R)(d, y)$Further, for states $(1 + \frac{\kappa}{R})Y^T - D < 0$, $0 \leq C_m(S) < c^*(C_m; \beta, \kappa, R)(s^e)$. Further, assuming an asymptotic positive marginal utility of tradeables consumption in Assumption 1, $A^*(c_{\max}; \beta, \kappa, R)(d, y) < c_{\max}$. Then the result on computing the least and greatest RCE in the theorem follows from Dugundji and Granas ([32], p.15). ■

Proof of Theorem 10

Proof. Notice the order continuity of the operator $A^*(C; \beta, \kappa, R)(s^e)$, it follows from a directly application of the main result in Balbus, et. al. ([9], Proposition 2). ■

Proof of Theorem 11

Proof. With the addition of Assumption 3 to Assumptions 1(a-e), 1(g), and 2, we can redefine the RCE fixed point problem into a single-step operator and then construct the RCE as in step 5 of the proof of Lemma 7. In particular, in equation 35, recalling the definition of the space $\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma) \subset \mathbf{C}^p_{++} = \{c \in \mathbf{C}^p | c(d, y) > 0\}$ in the proof of Lemma 7, for $c \in \mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$, define

$$C(c, C^T)(d, y, S) = C(c, c^{T*}_c(d, y)) = \inf\{c(d, y), c^{T*}_c(d, y)\}$$

where $c_c^{T*}(d,y)$ is given by equation 71. Then in the definition of our fixed point operator $A(c,C^T)(s^e)$ in equation 42, set the operator $A_c(C^T)(S) = c_c^{T*}(d,y)$ in equation so our fixed point operator $A(c,C^T)(s^e)$ under Assumption 3 now simplifies to the following

$$\begin{aligned} A(c,c_c^{T*})(s^e) &= \inf\{A_{uc}(c,c_c^{T*})(s^e), c_c^{T*}(d,y)\} \text{ when } c > 0 \\ &= 0 \text{ else} \end{aligned} \tag{71}$$

where in the constrained states, the RCE tradables consumption is $c_c^{T*}(d,y)$ and unique.
The operator $A(c,c_c^{T*})(s^e)$ has a unique strictly positive fixed point $c^{T*}(s^e)$ via the application of the Li-Stachurski version of the contraction mapping theorem discussed in step 5 in Lemma 7 noting as before $\mathbf{C}_{++}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e;\varsigma) \subset \mathbf{C}_{++}^p = \{c \in \mathbf{C}^p | c(d,y) > 0\}$ is a complete metric space under the norm

$$\rho(c_1,c_2) = \| u'_\varsigma \circ c_1 - u' \circ c_2 \|$$

where $\| u' \circ c_1 - u' \circ c_2 \| < \infty$, where $u'(c) = U'(A(c))A_1(c^T, y^N)$ is strictly decreasing in c^T under Assumption 1, and give $\mathbf{C}_{++}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e) \subset \mathbf{C}_{++}^p(\mathbf{D}^e \times \mathbf{Y}^e)$ its relative distance structure. So the unique strictly positive RCE tradables consumption is

$$c^*(d,y) = \inf\{c^{T*}(d,y), c_c^{T*}(d,y)\}$$

Finally, the contraction mapping theorem implies global stability of iterations. That is, we have for any initial $c \in \mathbf{C}_{++}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e;\varsigma)$,

$$A^{*n}(c)(d,y) = c^{T*}(d,y)$$

the unique consumption of tradables in the unconstrained states, and

$$\begin{aligned} c^*(d,y) &= \inf\{A^{*n}(c)(d,y), c_c^{T*}(d,y)\} \\ &= \inf\{c^{T*}(d,y), c_c^{T*}(d,y)\} \end{aligned}$$

■

Proofs for Section 4

Proof of Theorem 15

We will prove the theorem using several preliminary lemmas. First, it will be shown that equation (68), when it holds with equality, generates a sequence of increasing level of debt d_+ for any d as long as $y^T = y_{lb}$ if y_{lb} is sufficiently small. Then, using this result, we show that starting from any initial condition, the collateral constraint will bind in finite time. This lemma will be useful to show the existence of an accessible atom in the third lemma. Then, the forth lemma show the existence of a unique invariant ergodic measure.

From now on we will assume that assumption 1 holds. Additionally, assume that in any SCE we have $d_{t+1} \leq H$ with $H \equiv \kappa(y_{ub} + P_{ub}y^N)$. Lemma 17 will show that, once the collateral constraint is imposed, H will not bind in equilibrium.

Lemma 17 *In equilibrium when $R\beta < 1$, we have $d_{++} > d_+ > d$ for any $d \in K_2$ if the collateral constraint does not bind, $y^T = y_+^T = y_{lb}$ and $y_{lb} \in (0,\epsilon)$ with $\epsilon > 0$.*

Proof. Note that in equilibrium, WLOG it is possible to write $U'(A(c_1,c_2))A_1(x_1) \equiv u'(y^T - d + R^{-1}d_+)$. Then, under the assumptions stated in the lemma, it is clear that equation (68) can be written as:

$$u'(y^T - d + R^{-1}d_+) = R\beta \sum_{y_+^T} u'(y_+^T - d_+ + R^{-1}d_{++})q(y_+^T)$$

Where $d_{++} \in K_2$. Suppose, to generate a contradiction, $d \in K_2$ and $d_+ \leq d$. Then, as $R > 1$ for ϵ sufficiently small, we have $u'(y^T - d + R^{-1}d_+) \longrightarrow u'_{ub}$, where u'_{ub} can be constructed using the definition of u' together with assumptions $1 - a1$ and $1 - c$ and $1 - f$. Then, as $R\beta < 1$, $u'_{ub} > R\beta \sum_{y_+^T} u'(y_+^T - d_+ + R^{-1}d_{++})p(y_+^T)$ which implies that d_+ is not optimal. Then, we must have $d_+ > d$ as desired. Replacing d with d_+, we get $d_{++} > d_+$. ∎

Lemma 18 *For any $z \in Z$ the sequence $\{z, \phi_1, \phi_2, ...\}$, generated by (Z, P_φ), will hit the collateral constraint in finite time.*

Proof. Take any $y_0^T \in Y$, $d_0 \in K_2$ and a sequence with τ elements in $Y \times Y \times ... \times Y$ with $\{y_0, y_{lb}, ..., y_{lb}\}$. Then, the results in section 3 imply that, as long as the collateral constraint does not bind, $d_{\tau+1}(y_\tau^T, y_{lb}, ..., y_{lb}, y_0; d_0) = d'(y_\tau^T, d)$ and $P_\tau(y_\tau^T, y_{lb}, ..., y_{lb}, y_0; d_0) = P(d'(y_\tau^T, d))$, where the equalities follow from applying iteratively backwards the minimal state space policy function on equations (49) and (50) together with the envelope theorem, both derived in section 3. Note that the dependence of P on y^N has been omitted. Further, if the collateral constraint does not bind, we know from section 3 that $d'(y_\tau^T, d)$ / $P(d'(y_\tau^T, d))$ is decreasing /increasing in y_τ^T for each d. Further, $d'(y_\tau^T, d)$ / $P(d'(y_\tau^T, d))$ is increasing /decreasing in d for each y_τ^T. Lemma 20 below formally proves these claims. Then, using Lemma 17 we know that $\{d_1, ..., d_{\tau+1}\}$ / $\{P_0, ..., P_{\tau-1}\}$ is a (strictly) increasing / decreasing sequence which in turn implies that $g_t(y_t^T, y_{lb}, ..., y_{lb}, y_0; d_0) \equiv \kappa(y_t^T + P_t y^N) - d_{t+1}$ is a strictly decreasing sequence in t. To complete the proof we must show that: i) there exists a $y_\tau^T \in Y$ such that $g_\tau \leq 0$ and ii) $\tau < \infty$.
i) Suppose the collateral constraint does not bind. Then, $d_{t+1} \longrightarrow H$. By the definition of H and the fact that $d_{t+1} = d'(y_{lb}, d_t)$, we know that $|H - d_{t+1}| = H - d_{t+1} < \varepsilon$ for $t \geq N_\varepsilon$. For any given κ, we can take $\varepsilon \equiv \kappa(y_{ub} + P_{ub}y^N) - \kappa(y_{lb} + P_{ub}y^N) = \kappa(y_{ub} - y_{lb})$. Then, $d'(y_{lb}, d_t) > \kappa(y_{lb} + P_{ub})$, which is a contradiction. Then, the collateral constraint binds. That is, $g_\tau \leq 0$ for $g_\tau(y_{lb}, y_{lb}, ..., y_{lb}, y_0; d_0)$
ii) Simply take $\tau = N_\varepsilon$.
Note that H can be defined for any $y^T > y_{lb}$ which in turn implies that ε can be assumed to be arbitrarily small as desired. As the initial conditions were arbitrary, the proof is completed. ∎

Before stating and proving the next lemma we need some additional notation. Let $z \in Z$. Then, any solution to the system defined by equations (52) to (56) will be denoted $z(d, y^T) \equiv z = [d \quad y^T \quad y^N \quad c_1 \quad c_2 \quad p \quad m]$.

Lemma 19 *Let $J_1 \equiv Z$, where Z was defined in section 4.1. There is a point $d_* \in K_2$ with $d_{ub} > d_* > d_{lb}$ and a selection $\varphi \in \Phi$, where Φ is the equilibrium correspondence which contains all Generalized Markov Equilibria, such that for any $(y_0^T, d_0) \in Y \times K_2$, there is a sequence $\{\phi_0, \phi_1, \phi_2, ...\}$, generated by (J_1, P_φ), which satisfies $\phi_\tau = z(d_*, y_{lb}) \in J_1$ with $\tau < \infty$.*

First, some notation and a auxiliary lemma. Let $d'(d_0, y_{lb})$ be the policy function obtained from solving the optimization problem in the RE defined in section 3 for the unconstrained case. It can be seen from the results in section 3 that, as long as we are dealing with the unconstrained problem, $d'(d_0, y_{lb})$ is independent of prices and, thus, we can take this policy function for any c^T. Then, $\varphi \in \Phi$ satisfies:

$$d'(d_*, y_{lb}) = \kappa \left[y_{lb} + y^N \left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(d_*, y_{lb})/R) - d_*)} \right) \right] \tag{72}$$

$$U'\left\{A_1\left(y_{lb}+(d'(d_*,y_{lb})/R)-d_*\right)\right\}=\beta RE_{\varphi}[-d'(d_*,y_{lb})] \tag{73}$$

Where φ is defined by taking any vector $d''(y')\in K_2$ for any $y'\in Y$ such that:

$$U'\left\{A_1\left(y_{lb}+(d'(d_*,y_{lb})/R)-d_*\right)\right\}=\beta R\sum_{y'}[U'\left\{A_1\left(y'+(d''(y')/R)-d'(d_*,y_{lb})\right)\right\}]q(y')$$

Before proving Lemma 19, we need a preliminary lemma. It simply proves that, in an unconstrained framework, in partial equilibrium, more "disposable income" means more consumption and less debt.

Lemma 20 *Let $y^D\equiv y^T-d$ and $c_1\left(y^D\right)$ be a tradable optimal unconstrained consumption. Then, in any unconstrained equilibrium, $y^D>\widetilde{y}^D$ implies i) $c_1\left(y^D\right)>c_1\left(\widetilde{y}^D\right)$ and ii) $d'\left(y^D\right)<d'\left(\widetilde{y}^D\right)$*

Proof. i) Suppose not. Then, $y^D>\widetilde{y}^D$ and $c_1\left(y^D\right)\le c_1\left(\widetilde{y}^D\right)$. Then, the budget constraint in any unconstrained equilibrium implies $d'\left(y^D\right)<d'\left(\widetilde{y}^D\right)$. As $c_1\left(y^D\right)$ and $d'\left(\widetilde{y}^D\right)$ are optimal, we have:

$$U'\left(A_1(y^D)\right)\ge U'\left(A_1(\widetilde{y}^D)\right)=\beta RE\left(-d'\left(\widetilde{y}^D\right)\right)>\beta RE\left(-d'\left(y^D\right)\right)$$

Which implies a contradiction as $c_1\left(y^D\right)$ and $d'\left(y^D\right)$ are assumed to be optimal.

ii) Let $y^D>\widetilde{y}^D$ and $c_1\left(y^D\right)>c_1\left(\widetilde{y}^D\right)$. Assume, in way of contradiction $d'\left(y^D\right)\ge d'\left(\widetilde{y}^D\right)$. Then:

$$U'\left(A_1(y^D)\right)<U'\left(A_1(\widetilde{y}^D)\right)=\beta RE\left(-d'\left(\widetilde{y}^D\right)\right)\le\beta RE\left(-d'\left(y^D\right)\right)$$

Which implies a contradiction as $c_1\left(y^D\right)$ and $d'\left(y^D\right)$ are assumed to be optimal. ■

Lemma 21 *In any unconstrained equilibrium, there is a decreasing sequence of debt with increasing tradable consumption for the same level of tradable output, y^T*

Proof. Let $d'(d_t,y_t^T)=d_{t+1}$. Thus, $U'\left(A_1(y_t^T-d_t+(d_{t+1}/R))\right)=\beta RE\left(-d_{t+1}\right)$. Take $\widetilde{d}_t<d_t$. Then $U'\left(A_1(y_t^T-\widetilde{d}_t+(d_{t+1}/R))\right)<\beta RE\left(-d_{t+1}\right)$. Then, there exist $\widetilde{d}_{t+1}<d_{t+1}$ with $(\widetilde{d}_{t+1}/R)-\widetilde{d}_t>(d_{t+1}/R)-d_t>0$ such that $U'\left(A_1(y_t^T-\widetilde{d}_t+(\widetilde{d}_{t+1}/R))\right)=\beta RE\left(-\widetilde{d}_{t+1}\right)$. By letting $d_{t+1}=\widetilde{d}_t$, $\widetilde{d}_{t+1}=d^*_{t+2}$ and $\widetilde{d}_t=d^*_{t+1}$, $d_t=d^*_t$, we obtain the decreasing sequence $\left\{d^*_{t+i}\right\}_i$ which generates a increasing consumption sequence $\left\{c^*_{1,t+i}\right\}_i$ because $(\widetilde{d}_{t+1}/R)-\widetilde{d}_t>(d_{t+1}/R)-d_t>0$ and $y^T_{t+i}=y^T$ for all i. ■

Lemma 22 *The selection $\varphi\in\Phi$ exists.*

Proof. Lemma 20 implies that there exist $d_*\in K_2$ such that:

$$d'(d_*,y_{lb})=\kappa\left[y_{lb}+y^N\left(\frac{A_2(y^N)}{A_1(y_{lb}+(d'(d_*,y_{lb})/R)-d_*)}\right)\right]$$

This is possible as the LHS / RHS of this equation is increasing / decreasing in d, both sides are continuous functions of d according to the results in section 3.2 and

$$d'(0,y_{lb})<\kappa\left[y_{lb}+y^N\left(\frac{A_2(y^N)}{A_1(y_{lb}+(d'(0,y_{lb})/R))}\right)\right]$$

That is the collateral constraint does not bind if debt is non-positive.

Take any $y_0^T, d_0 \in J$ and a sequence $\{y_0^T, y_{lb}, ..., y_{lb}\}$ of $\tau + 1$ elements. The results in Lemma 17 imply that, as long as the collateral constraint does not bind in period $\tau < \infty$ (this assumption can be imposed WLOG due to Lemma 18), there is a constant sequence of tradable consumption $\{c_{1,0}, ..., c_{1,\tau-1}\}$ with $c_1(y_0^T, d_0) = c_{1,t}$ for $0 \leq t \leq \tau - 1$ which can be implemented as a SCE. Further, as it is shown in Lemma 20, it is possible to choose $c_1(y_0^T, d_0)$ to be decreasing in d_0 and increasing in y_0^T. Equipped with these paths we will deal with a fraction of all possible initial conditions in $Y \times K_2$. In order to deal with the rest of the space we will use 17 and the inequality nature of the Euler equation to deal with the constrained case, if necessary.
Now take $y_0^T = y_{lb}$ and $d_0 = d_*$ with $\kappa(y_{lb} + P(x_1(y_{lb}, d_*))y^N) = d'(y_{lb}, d_*)$. The existence of d_* follows from Lemmas 19 and 22. This point defines the "atom" and we must show that there exist a positive probability sequence starting from any initial condition that hits it in finite time. Intuitively, the atom will be defined as the level of wealth, d, for which the collateral constraint binds with equality at $t = 0$ for the lowest possible level of current income, y_0^T. Thus, the strategy of the proof is to show that regardless of the initial condition, it is possible to construct a positive probability path that will hit the constraint: a) later, b1) with a bigger tradable consumption level and with more debt, both today or b2) with more debt tomorrow (and not necessarily with more consumption today). Call this last point $z(d_\tau, y_\tau)$. First, note that, as the collateral constraint hits with equality in the atom, for any other path, we will have more debt but not necessarily more consumption. In any case, the euler inequality implies that $z(d_*, y_{lb})$ satisfies the system of equations which defines the GME due to the strict inequality in the primal formulation of this equation. That is, the qualitative properties of $z(d_\tau, y_\tau)$ imply, due to the inequality in the Euler equation in the primal characterization of the sequential equilibria coupled with the backwards nature of the definition of any GME, that $z(d_\tau, y_\tau)$ and $z(d_*, y_{lb})$ are both a solution to the constrained system of equations which define the GME. This last claim is proved in Lemma 22.
We will now show that the chain will hit the atom, $z(d_*, y_{lb})$, starting from any initial condition in $J = Y \times K_2$. We will proceed in 2 regions: i) $d_0 < d_*$ and $y_0^T = y_{lb}$, ii) $d_0 > d_*$ and $y_0^T > y_{lb}$. Intuitively, region i) insures, due to Lemma 20, that the collateral constraint will not bind at $t = 0$ and that initial tradable consumption is bigger than the atom level. Using Lemma 17 and 18 we will construct a positive probability sequence with increasing debt which insures the "reversion" to the atom at the time the path hit the constraint. Region ii) has 2 possible sub-regions. ii.a) $y_0^T - d_0 > y_{lb}^T - d_*$, in which case again due to Lemma 20 we will have bigger initial consumption and smaller debt when compared with the atom level. Thus, we can construct a sequence with increasing debt in an unconstrained environment using the arguments in region i) until we exceed the atom's level and revert to it when the path hit the collateral constraint. ii.b) $y_0^T - d_0 < y_{lb}^T - d_*$, in which case the consumption level is smaller when compared to the atom level due to Lemma 20. Thus, we must construct an increasing consumption sequence in order to revert to the atom. In this case there are multiple possibilities: the constraint can be bigger (ii.b.1) or smaller (ii.b.2) compared with the atom level. In region ii.b.1 an increase in consumption will take place in a constrained environment if p is not sufficiently sensitive or elastic (ii.b.1.1). Contrarily, if prices are sufficiently elastic, an increase in consumption will take place in an unconstrained environment (ii.b.1.2). Finally, in region ii.b.2, when the constraint is bigger with respect to the atom level, as debt is also bigger we may have a constrained or an unconstrained regime. In these cases we can use the results in regions ii.b.1 to generate an increasing consumption level until we reach the atom consumption.
i) For $d_0 < d_*$ and $y_0^T = y_{lb}$, we have $\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_0))y^N)$ and $d'(y_{lb}, d_0) < d'(y_{lb}, d_*)$ from Lemma 20 which, using Lemma 18, implies that for the sequence $\{\phi_0, ..., \phi_t\}$ the chain will hit the collateral constraint in $t = \tau > 0$ with $d_\tau > d_*$.
We claim that the system of equations given by (52) to (56) can also be solved by $z(d_*, y_{lb})$ and thus we have that $\{\phi_0, ..., z(d_*, y_{lb})\}$ is an equilibrium trajectory. To prove this claim, note that from the definition of an equilibrium correspondence, we have that any d in z with $z \in Z$ that solves $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_0))y^N) - d)\} \geq E(m_+)$ is a predecessor of $z_+(y_+^T)$ for any $y_+^T \in Y$. As $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} > E(m_+)$, equations (52) to (56) imply that $\{\phi_0, ..., z(d_*, y_{lb})\}$ is an equilibrium trajectory as desired.
ii) For $d_0 > d_*$ and $y_0^T > y_{lb}$.

Region ii.a): $y_0^T - d_0 > y_{lb} - d_*$. As y^D is bigger in this case when compared with the atom level and we have more tradable output, $d'(y_0^T, d_0) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < \kappa(y_0^T + P(c_1(y_{lb}, d_0))y^N)$ and we can use the arguments in region i).
Region ii.b): $y_0^T - d_0 < y_{lb} - d_*$.
Region ii.b.1): $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$. That is, the collateral hits at $t = 0$ with $c_1(y_{lb}, d_0) > c_1(y_0^T, d_0)$ which implies that we must generate an increasing sequence of consumption from the constrained regime. Note than when we increase tradable consumption, depending on the sensibility of p with respect to c^T, we may enter into a constrained (ii.b.1.1) or an unconstrained regime (ii.b.1.2).
Region ii.b.1.1): $1 \geq \kappa y^N R^{-1} p'$, where p' is the derivative of (52) with respect to d_+. As $c_1(y_{lb}, d_*) > c_1(y_0^T, d_0)$ and

$$\begin{aligned}\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) &< \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) \\ &< d'(y_0^T, d_0)\end{aligned}$$

we have:

$$\begin{aligned}U'\{A_1(y_0^T + R^{-1}\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) - d_0)\} &> U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} \\ &= E(m_+; -\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)) \\ &> E(m_+; -\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N))\end{aligned}$$

The above inequality implies that any path in this region with $P(c_1(y_t^T, d_t)) < P(c_1(y_{lb}, d_*))$ is optimal. By setting $\{y_0^T, y_{lb}, ..., y_{lb}\}$ we can construct an increasing sequence $\{P(c_1(y_0^T, d_0)), P(c_1(y_{lb}, d_1)), ..., P(c_1(y_{lb}, d_\tau))\}$ converging to $P(c_1(y_{lb}, d_*))$ as desired.
Region ii.b.1.2): $1 < \kappa y^N R^{-1} p'$. In this case an increase in consumption take us to the unconstrained region. Using Lemma 21, we can generate a path $\{y_0^T, y_1^T, ..., y_\tau^T\}$, with $y_t^T > y_{lb}$; $t = 1, ..., \tau$, of increasing consumption and decreasing debt until we get $c_1(y_{lb}, d_*) < c_1(y_\tau^T, d_\tau)$. Then, set $\{y_\tau^T, y_{lb}, ..., y_{lb}\}$ The argument in region i) insure that there is a finite time, $\tau + \tau_1$, such that the path hits the atom.
Region ii.b.2): $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) > \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$. In this case, we can be either in the unconstrained or constrained case. For the former, we can use the same arguments as in section ii.b.1.2) as we need a path of decreasing debt and increasing consumption. For the latter, however, note that by setting $\{y_0^T, y_{lb}, ..., y_{lb}\}$ we enter into the ii.b.1) region for $t > 0$ as $\kappa(y_{lb} + P(c_1(y_{lb}, d_t))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)$ as $d_t > d_*$. ■

Lemma 23 *the results in Lemmas 17 to 19 imply that there exists a selection $\varphi \sim \Phi$ and a Markov process (J_1, P_φ) that has an accessible atom, $z(d_*, y_{lb})$, and is $P_\varphi(z(d_*, y_{lb}), .)-$irreducible*

Proof. Follows directly from proposition 1. ■

Lemma 24 *Let (J_1, P_φ) be the process defined in Lemma 19. If the collateral constraint hits at time $\tau > 0$ with $d_\tau > d_*$ and $c_{1,\tau} > c_1(y_{lb}, d_*)$ or with $d_{\tau+1} > d(d_*, y_{lb})$, then $\{\phi_0, \phi_1, ..., \phi_{\tau-1}, z(d_*, y_{lb})\}$ is an equilibrium trajectory.*

Proof. If the collateral constraint binds for consumption $c_{1,\tau}$ and debt d_τ, then it must satisfy $U'\{A_1(y_\tau^T + R^{-1}\kappa(y_\tau^T + P(c_{1,\tau})y^N) - d_\tau)\} \geq E(m_+)$. The conditions in the remark imply that $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_{1,\tau})y^N) - d_*)\} > E(m_+)$ as desired. Next note that

$$\begin{aligned}U'\{A_1(y_\tau^T + R^{-1}d_{\tau+1} - d_\tau)\} &\geq U'\{A_1(y_\tau^T + R^{-1}d(d_*, y_\tau^T) - d_*)\} \\ &= E(m_+(d(d_*, y_\tau^T))) \geq E(m_+(d_{\tau+1})).\end{aligned}$$

■

Lemma 25 *Let (J_1, P_φ) be the Markov process in Lemma 19. Then, (J_1, P_φ) has a unique, ergodic, invariant probability measure.*

Proof. Note that Lemma 19 imply that $P^{\tau}_{\varphi}(z(d_*, y_{lb}), \{z(d_*, y_{lb})\}) > 0$ with $\tau < \infty$. Given the results in Remark 4.2.1, proposition 4.2.2, theorem 8.2.1 and theorem 10.2.1 in Meyn and Tweedie ([53]) imply that (J_1, P_φ) has an unique invariant measure. As $\tau < \infty$ for any initial condition in J_1, theorem 10.2.2 in Meyn and Tweedie ([53]) implies that the invariant measure is a probability measure. As it is unique, the Krein-Milman theorem (See Futia, [34]) implies that this measure is ergodic. ■

Online Appendix

This section contains additional details that complements the body of the paper.

Supplementary material for section 2

This section characterizes the sequential competitive equilibrium when $\beta R < 1$ with Y finite. In this paper, we do not discuss existence issues for the case that $\beta R \geq 1$. In a related paper, Pierri and Reffett ([64])) identify sufficient conditions for extending the results to this case (including setting with more general shocks and continuous shock spaces). The critical complication for introducing $\beta R \geq 1$ is obtaining compactness when studying the long-run behavior of such a model relative to stationary equilibrium. Turns out, if we endow the model with a satiation point, the equilibrium has a degenerate steady state as consumption converges to a Dirac measure a.e. It is possible, as in Hansen and Sargent ([38]), to allow for a generalization of this last type of equilibrium by assuming that the satiation point (called "bliss point") is a random variable. Although it can be useful in some applications (i.e. asset pricing with no trading, etc.), this type of equilibrium has really restrictive dynamics. Taking into account the question at hand, we defer the discussion of this case to a separate paper.
An example of the restriction on preferences implied by Assumption 1 can be seen in the table below. Note that we actually prove the existence of the RCE (which is a SCE) for case 1 and 3.

Pref.	Does SCE Exist?	$c > 0$?	MU bounded?	Are Homothetic?
CD	Unknown	YES	NO	YES
LOG	Unknown	YES	NO	YES
CES	Unknown	YES	NO	YES
Mod. CD	Theorem 1	YES	YES, above	NO
Mod. LOG	Theorem 1	YES	YES, above	NO
Mod. CES	Theorem 1	YES	YES, above	NO
Mod. CES 2	Theorem 1	YES	YES, AF zero	NO

Table 1: Restriction on preferences

where the abbreviation "MU" stands for "marginal utility" and "AF" for "away from" . Below we provide a concrete parametrization of preferences for each of the cases presented in table 1. Note that Theorem 4 requires MU to be bounded above and away from zero. The requirement that MU is bounded above breaks the homotheticity of preferences (i.e., Inada conditions), but the assumption of bounded away from zero allows for the homothetic case. As we will later show, the existence of minimal state space recursive equilibrium can be shown even without an finite upper bound on marginal utility; but the existence of a maximal consumption still requires MU of consumption be bounded away from zero.
More to the point, the proposed utility functions can be made arbitrarily close to their homothetic counterpart and the numerical section in this paper, which suggests that assuming boundedness, instead of imposing restrictions on marginal utility, and using standard CES preferences works in practice. We

need bounds on marginal utility to ensure the compactness of the equilibrium set in order to complete the existence proof of the SCE. As the numerical results in this paper use extensively the quantitative implications of this last type of equilibrium, our simulations in this paper in section 5 are well behaved under the standard CES preferences.[53]

Finally, this assumption on boundedness of MU in Assumption 1.f can be replaced with standard Inada conditions in the case proving the existence of RCE as we show in the next section of the paper. That is, RCE stochastic dynamics are much simplier and their characterization can can be greatly sharpened using Euler inequality methods. In particular, guaranteeing the uniform interiority of tradeable consumption can be done directly when constructing the existence of RCE. We should mention, the existence of RCE under Assumption 1.f can also be proved (i.e., standard Inada conditions are not needed for the existence of RCE). What is needed is the existence of a sufficiently high MU's of tradeables consumption near zero tradeables consumption to guarantee stochastic equilibrium dynamics do not (almost everywhere) converge to 0. Further, RCE are, of course, a subset of SCE. But as in quantitative work, it appears that stochastic dynamics generated under Assumption 1.f vs. Inada conditions are not different, we do consider the question of the existence of SCE under the boundedness of MU conditions in Assumption 1.f so we allow for the possibility that SCE exist that need to be RCE.

Examples of utility functions in table 1

This section contains concrete examples of utility function which ensure that marginal utility is bounded.

$$\begin{gathered} Mod.\quad CD \quad (c_1+\gamma)^{\alpha}(c_2+\gamma_1)^{\beta} \\ u: X \to \Re, \quad X \supseteq \Re^2_+, \quad \gamma, \gamma_1 > 0, \quad c \in X \Rightarrow c + [\gamma, \gamma_1] > 0 \end{gathered}$$

The "Mod. CD" preferences are defined over a consumption set which includes the "zero"-vector and γ insures that marginal utility remains bounded above over the entire consumption set, X. The "Mod. LOG" and "Mod. CES" are similar. Just replace $(c_1+\gamma)^{\alpha}\ (c_2+\gamma_1)^{\beta}$ by $ln\,(c_1+\gamma) + ln\,(c_2+\gamma_1)$ and by $(a\,(c_1+\gamma)^{\alpha} + a\,(c_2+\gamma_1)^{\alpha})^{(1/a)}$ respectively with $a > 0$. The "Mod. CES 2" are rather different as they are intended to keep MU bounded away from zero. In particular,

$$\begin{gathered} Mod.\quad CES\ 2 \quad (a_1\,(c_1)^{(1-\alpha)} + (1-\alpha)a_2c_1 + b_1\,(c_2)^{(1-\alpha)} + (1-\alpha)b_2c_2)^{(1/(1-\alpha))} \\ u: X \to \Re, \quad X \supseteq \Re^2_+, \quad \alpha > 1, \quad a_1, a_2, b_1, b_2 > 0 \end{gathered}$$

If we combine "Mod. CES" with "Mod. CES 2" we insure that MU remains bounded above and away from zero, which in turn guarantees that p is positive and finite. These are sufficient conditions for the existence of an a.e. compact equilibrium.

Supplementary Material for Section 5.2

In order to continue with the description of possible multiple equilibria we must incorporate 2 restrictions. The first comes from $P > 0$ as figures 1 and 2 don't guarantee that $P1 > 0$, even tough any meaningful equilibrium must have positive prices (i.e., $P^* > 0$). From equation (61) it is clear that if $P = 0$, then $d = y_{lb}(1+\kappa/R) \equiv d_{P=0}(y_{lb})$ and $d < d_{P=0}(y_{lb})$ implies $P > 0$. Thus, $d_{P=0}(y_{lb})$ is the upper bound on debt, d_{ub}. Further, we must guarantee that consumption remains positive. Using equation (61) again we can see that the locus, on the plane (d, P), for which consumption equals 0 is given by:

[53]Non-homotheticity of preferences have been used recently in many strands of the macroeconomics literature. For example, Rojas and Saffie [66] recently has explored the importance of non-homothetic preferences in models of sudden-stops. In the context of precautionary savings models, Straub ([77]) found that under homothetic preferences consumption is linear in permanent income, a fact which is at odds with data. He extends the canonical precautionary savings model with heterogeneous agents to include non-homothetic preferences.

$$d = y^T + (\kappa/R)[y^T + Py^N] \tag{74}$$

Assume $\kappa/R = 1$. Clearly, in the (d, P) plane, equation (74) is a linear function $P = dR - 2y^T R$. Moreover, $P(y^T) \equiv -2y^T/y^N = -2y^T R$, where the second equality follows from $y^T R = 1$ which was imposed above, defines the intersection of this locus with the P-axis (i.e. the value for P that generates $c^T = 0$ with $d = 0$) and $d_2(y^T) \equiv 2y^T$ with the d-axis (i.e. the value for d that generates $c^T = 0$ with $P = 0$) [54]. Above this locus lie all the pairs (d, P) for which $c^T > 0$. Applying the implicit function theorem to (61) for $P > 0$, we get:

$$\frac{DP}{Dd} = -1\left(y^N[1/2P^{-1/2} - \kappa y^N]\right)^{-1} < 0 \text{ if } P < \left[2\kappa y^N\right]^{-2} \text{ and } \frac{DP}{Dd} > 0 \text{ if } P > \left[2\kappa y^N\right]^{-2}.$$

Remarkably, note at point the $P^* = \left[2\kappa y^N\right]^{-2}$, we get $DP/Dd = -\infty$. This point can be used in (61) to obtain $f(P^*) = y^{T*}(R+\kappa) = K(0)$, the value for y^T which generates P^* in (61) when $d = 0$. Any $y^T < y^{T*}$ will generate a discontinuity in equation (61) in the (d, P) plane for $d > 0$. We call this point E. Figure 3 combine figures 1 and 2 with the 2 restrictions mentioned above.

For expositional purposes we have replaced y_{lb} with $Ymin$, $y_{ub} > y^T > y_{lb}$ and $y_{ub} = Ymax$. Above the line joining the points $d_2(Y)$ and $P(Y)$, to the right of vertical axis and over the locus formed by $f(p) = K(d)$ we can find the admissible real exchange rate. If $y^T = Ymin$ and y^T is close to 0, the set of debt levels that admit multiple equilibrium is negligible. However, when $y^T = y^{T*}$, this set is formed by all debt levels between 0 and $d_2(y^{T*})$.
Summing up, equations (61) and (74) gives the admissible pairs (d, P) in figure 3. Note that $P(y^T)$ / $d_2(y^T)$ are all functions of y^T. Thus, an increase in this variable will shift both locus to the right, as depicted in figure 3. Thus, the set of admissible equilibria increases along with y^T as can be seen by looking at the region formed by points $A - B$ on the P-axis. That is, between 0 and d_2 there are 2 admissible exchange rates for each debt level d.

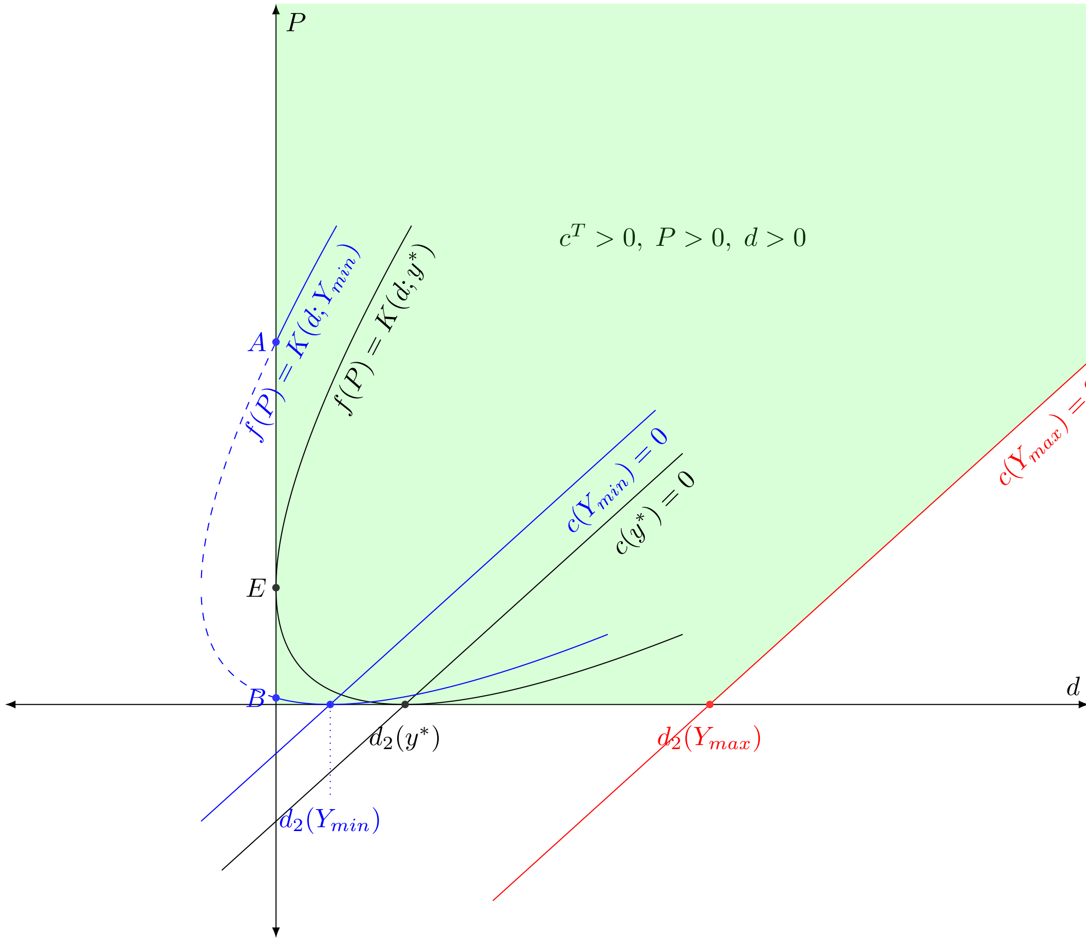
The last figure completely describes multiplicity. With respect to figure 3, we added the locus h. The claim above shows that: i) below h we can find the optimal pairs (d, p), ii) there are at least 2 optimal contours which insure the existence of multiple equilibrium for at least 2 shocks.
Up to now, in this sub-section, we have been silent about optimality. Assume that for some exchange rate level, P_0, there is a debt level d_0 which satisfies equation (56) for $y^T = y_{lb}$. That is, d_0 satisfies: $U'\{A_1(y_{lb} + R^{-1}\kappa\{y_{lb} + P_0 y^N\} - d_0)\} \geq E(m_+)$. Taking into account that we are assuming that (d, p) belong to a compact set, the existence of d_0 and P_0 follows without loss of generality given theorem 4 in section 2. Note that any $d = d_0 + (y^T - y_{lb})(1 + \kappa/R)$ will also satisfy equation (56). Now, the definition of $d_2(y^T)$ implies: $d_2(Y) - d_2(Ymin) = (Y - Ymin)(1 + \kappa/R)$. As can be seen in figure 3, if any d_0 is in the locus formed by $f(p) = K(d)$ (i.e. which satisfies equation (61)) with $(d, P) >> 0$, above the zero-consumption line $d_2(Ymin) - P(Ymin)$ line (i.e. which is above equation (74)) and satisfies equation (56), then $d = d_0 + (Y - Ymin)(1 + \kappa/R)$ is in the locus formed by (61) and above (74) but with the last equation crossing the d-axis at $d_2(Y)$. By replacing Y with $y^T > y_{lb}$ and $Ymin$ with y_{lb}, we know that d will also be feasible and optimal if d_0 is so: it will be to the right of the vertical axis, above the $d_2(y^T)$-$P(y^T)$ line, over the locus formed by $f(p) = K(d)$ (i.e. it will be feasible) and will satisfy equation (56) (i.e. it will be optimal). Finally, we must show that any the pair (d, P) satisfies the optimality requirement (i.e. it satisfies (56)) and find a contour in the plane in order to separate optimal from non-optimal pairs. We will start with (d_0, P_0). Take any $m_+(y^T_+)$ from Z which satisfies equation (55) with

[54] In this section we are assuming the SG-U preferences. That is, the relative price P from optimality is just:

$$p(C) = \frac{1-a}{a}\left(\frac{C^T}{Y^N}\right)^{1/\xi}$$

This implies that $P = 0$ if and only if $c^T = 0$, which why the value for d that generates both restrictions is the same, d_2.

Figure 3: Genericity of Multiple Equilibria: Feasibility and Intra-temporal optimality



The feasible set of any GME is depicted in this figure. The $P - d_2$ locus defines the "non-negative consumption region". The "x-axis" defines the debt region. The "y-axis" the positive prices region. Optimality according to the intra-temporal optimization condition in the constrained regime is represented by the oval contours. Note that for low tradable income levels, this last locus is "truncated" as can be seen in figure 1. There is a level of tradable income, y^T, for which the whole f locus contains prices in the constrained regime as $K(0) = R/2\kappa$. For bigger income levels, prices are not determined in the constrained regime and, thus, they do not belong to the $f = K$ locus.

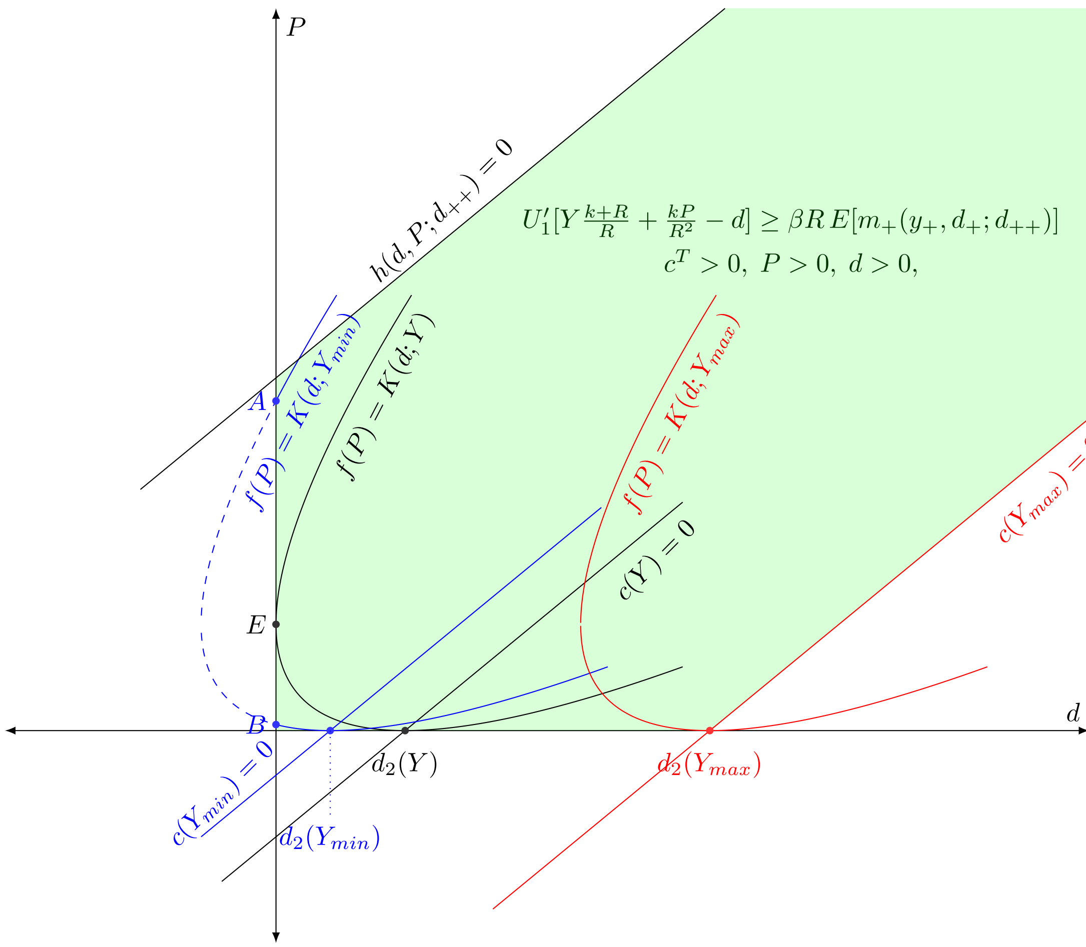
$d_+ = \kappa\{y_{lb} + P_0 y^N\}$ and any given d_{++}. Equation (75) below, when it holds with equality, defines a locus increasing in the (d, P) plane which determines optimality. Call this map $h(d, P) = 0$. Take any (d_0, P_0) over (75). Any $P \leq P_0$ is optimal because of the weak inequality in the primal optimization problem in the SCE. Thus, optimality lies below (75). Moreover, note that this locus in increasing in d_{++}. Thus, $h(d, P; d_{++}) = 0$ As can be seen from the definition of GME, $d_{++}(y_+)$ for each $y_+ \in Y$ defines a selection of the equilibrium correspondence and, thus, 1 of possible multiple GME. As the upper bound on d_{++} can be chosen freely [55]. Then, there is always a value in $[d_l, d_u]$, the set which contains d by assumption, which insures optimality. That is, which satisfies:

$$U' A_1 \left(y^T + R^{-1}\kappa\left(y^T + Py^N\right) - d\right) \geq \beta R E[m_+\left(-R^{-1}\kappa\left(y^T + Py^N; d_{++}\right)\right)] \tag{75}$$

Figure 4 illustrates these facts. Note that the multiple displacements of (61) in solid lines form the

[55] For expositional purposes, in this subsection we are assuming that endogenous variables lie in a compact set. Thus, the existence of an arbitrarily large upper bound can be insured without loss of generality. However, in section 2, we showed that it depends on the bounds on marginal utility (see Table 1).

Figure 4: Genericity of Multiple Equilibria: Feasibility and Optimality



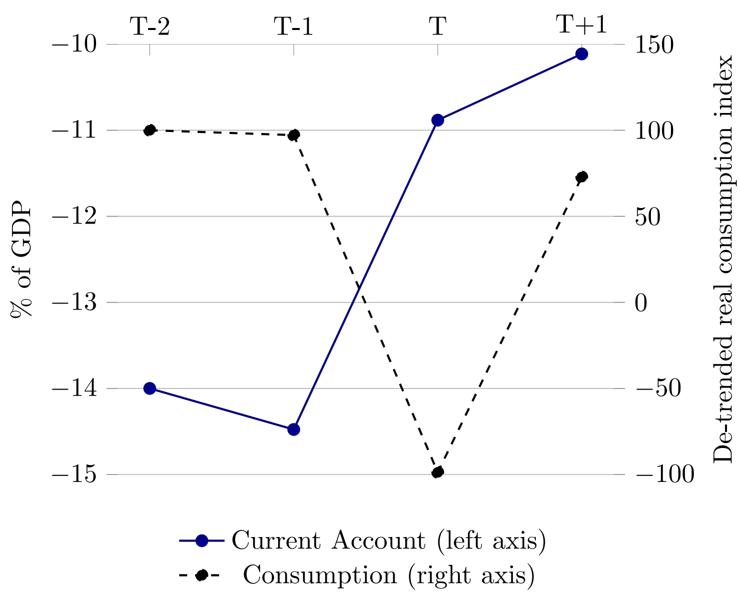
The h locus contains the (d, P) pairs which satisfy intertemporal optimality with equality in the constrained regime. Below it we can find the optimal (d, P). As h is increasing in $d_{++}(y_+)$, which represents the selection of the equilibrium correspondence Φ, we can choose h to be conveniently located. In the ergodic case, the equilibrium correspondence could be a truncated version of figure 4. Lemma 3 and theorem 4 insures that the intersection of the region below h and the region defined by the oval contours in the non-negative orthant formed by the (d, P)-axis is not empty.

equilibrium correspondence of some GME for an arbitrarily large selection $\varphi \in \Phi$. That is, figure 4 below presents the full picture as it combines the feasible pairs in figure 3 with the optimal ones in $h(d, P; d_{++}) = 0$. Note that there is a fraction of equation (61) which intersects with $d_2(Ymin) - P(Ymin)$ that is depicted in light grey. This is because it evolves $d < 0$. As the level of P becomes smaller, c^T does so and it is not necessary to take debt to satisfy optimality. In other words, the light grey areas represent the unconstrained regime for $y^T = y_{lb}$. A similar argument holds for $y^T = Ymax \equiv y_{ub}$. As it is shown in the appendix for section 4 (see Lemma 20), in the unconstrained regime, tradable consumption is increasing in tradable output, while debt is decreasing in the same variable. Thus, it is possible that $d(d, Ymax) < p(Ymax - d + d(d, Ymax)/R, y^N)$, where $p(.,.)$ is equation (49) and $d(.,.)$ is the policy function for the unconstrained regime associated with the envelope m_+. The locus formed by equation (61) which intersects with $d_2(Ymax) - P(Ymax)$ is depicted in light grey as the pair $(d+(Ymax-Ymin)(1+\kappa/R), PYmax)$, where $f(PYmax) = K(d+(Ymax-Ymin)(1+\kappa/R))$ may belong to the unconstrained regime given the fact that $Ymax$ can be chosen to be arbitrarily large. Further, note that there is a point for which $d(d, Y^T) = p(y^T - d + d(d, y^T)/R, y^N)$, with $Ymin < y^T \leq Ymax$ (see Lemma 18 in the appendix). Thus, as $p(.,.)$ is continuous and $d(.,.)$ are continuous and decreasing in y^T, and $c^T = Y^T - d + d(d, Y^T)/R$ is increasing in Y^T, we know that the collateral constraint will be binding for any $y < Y^T$. Thus, the desired result follows.

Supplemantary material for section 5.2.2

In this subsection we show that an ergodic GME, summarized by equations (52) to (56), is capable of generating 2 different types of sudden stops. At the same time, each of these events can be divided in 3 phases. i) The pre-sudden stop, characterized by an increase in the current account deficit. ii)The sudden stop itself, which consists of a sharp drop in consumption and a current account reversal. iii) The post-sudden stop. The events are mainly differentiated by phase iii). In the first type, exemplified by Portugal in 2008, there is a moderate recovery in consumption. In the second type, represented by Spain in 2008, consumption keeps falling after the sudden stop. Figures 5 and 6, borrowed from Pierri, et. al. ([63]) illustrate this situation.

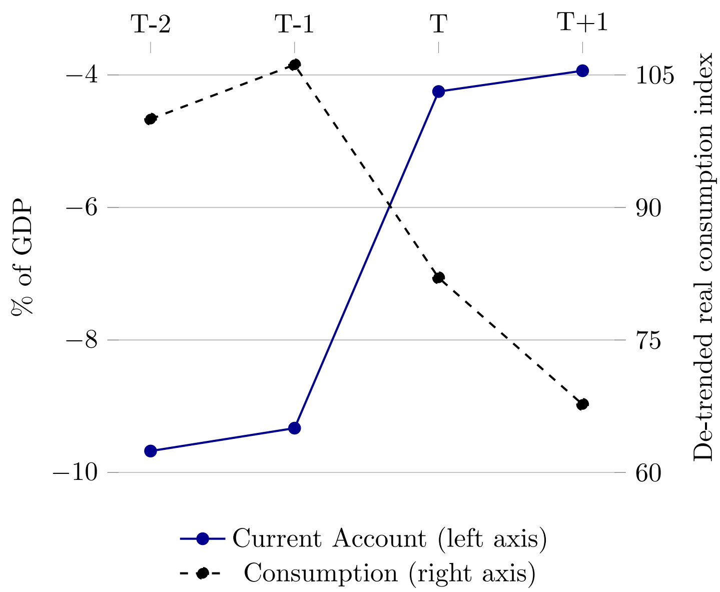
Figure 5: Anatomy of a Sudden Stop: Portugal 2008



The figure is constructed using the data set in Pierri, et. al. (2018). The authors refine the definition of a sudden stop to incorporate the 3 phases mentioned in section 5.2. The dotted line represents the current account divided by GDP, both at current prices. The left index is measured in porcentaje points. The right index, in full line, contains a consumption index constructed using the de-trended (by Hodrick and Prescott, HP) series at constant prices. The base of the index is 100 at "T-2", where "T" is the date of the sudden stop (2008).

Moreover, the GME is capable of simulating a "Fisherian deflation" (i.e. a path of real exchange rate depreciation coupled with falling consumption) and a sudden stop without relying on a large, unanticipated shock that impose the loss of access to capital markets by assumption. This shock is typically represented by a sudden change in a (non-stochastic) parameter or a change in the support of the distribution of exogenous shocks. Neither of these assumptions are required once we change the type of SCE we study from RE to more general GME representations. To our knowledge, this is the first attempt to show that a general equilibrium model with occasionally binding price dependent collateral constraints is capable of generating a collapse in borrowing followed by a "financial accelerator effect", caused by de-leveraging, and at the same time preserve the ergodicity of the dynamical system which, as will be shown in section 5.3, is essential for simulating and calibrating the model.

Figure 6: Anatomy of a Sudden Stop: Spain 2008



To model a sudden stop we must show the existence of an abrupt reversal in foreign borrowing. By "abrupt" we mean that it must be an intra-period event. This is the *same* effect caused by a sharp reduction in κ but without imposing an exogenous event (i.e. an event that can not be explained within the theoretical framework embodied in the model). In order to show the existence of such an event the multiplicity of equilibria will be essential.
Let "T" in figures 5 and 6 be the time of the sudden stop. Before that, in phase i), we are in the unconstrained regime. In the case of Portugal, we observe a small decrease in consumption (3%) and an increase in the current account deficit. By setting $y^T_{T-2} = y^T_{T-1} = y_{lb}$, as the unconstrained regime can be thought as a standard partial equilibrium concave savings problem, we can match the observed path [56]. In the case of Spain, we observe both an increase in consumption and a reduction in the current account deficit. The same arguments can be used to show that this path can also be generated in the unconstrained regime for a sequence of shocks which satisfies $y^T_{T-2} > y^T_{T-1} > y_{lb}$ [57]. Then, suppose that for some debt level d and $y^T = y_{lb}$, the collateral constraint is binding. This happens in period T without loss of generality [58]. Then, equation (56) implies:

$$U'\{A_1(y_{lb} + R^{-1}\kappa\{y_{lb} + py^N\} - d)\} \geq \; E(U'\{A_1(y^T_+ - \kappa\{y_{lb} + py^N\} + R^{-1}\overline{d})\}) \tag{76}$$

Where $\overline{d}$ defines the ergodic selection $\varphi \sim \Phi$ in theorem 1 and the right hand side of equation (76) follows from the existence of an envelope shown in section 3.

One of the most relevant aspects of a GME is that it is computed "backwards". That is, we are interested in finding d, instead of d_+ as in the RE It turns out that this fact can be used, combined with the primal version of the Euler equation (76), to obtain a sudden stop without requiring a jump in the exogenous parameters of the model. Let c, d_+ be the consumption and debt level tomorrow in (76). Note that, as we are in the constrained regime and P is increasing in tradable consumption, any $c_* < c$ will also satisfy (76). Take a pair (d_*, P_*) with $y_{lb} + py^N > y_{lb} + p_* y^N$, where $P(c^T) \equiv p > P(c^T_*) \equiv p_*$ with p and p_* defined

[56] The proof follows immediately from Lemma 17 and 20 in the appendix of section 4

[57] See Lemma 20 in the appendix for section 4.

[58] Lemma 18 in the appendix containing the proofs of section 4 implies that starting from any initial condition, the collateral constraint binds in finite time with positive probability.

by the intra-temporal optimality condition in the system defined by equations (52) to (56). Now, the discussion in the preceding section implies that there are 2 possible branches for the equilibrium selection. Taking the lower one, we know that p is decreasing in d as the branch has negative slope (see figure 4), which in turn implies $d < d_*$. Thus, $c = y_{lb} + R^{-1}\kappa\{y_{lb} + py^N\} - d > y_{lb} + R^{-1}\kappa\{y_{lb} + p_*y^N\} - d_* = c_*$ as desired. As equation (76) holds with inequality, this implies that the level of debt tomorrow is binding at p_*, that is $d_{+,*} = \kappa\{y_{lb} + p_*y^N\}$.
The discussion above suggests that it is possible to set the level of next period borrowing such that it is binding for the new level of consumption c_* as equation (76) is allowed to hold with strict inequality. Remarkably note that there exist a $\kappa' < \kappa$ such that $\kappa'\{y_{lb} + py^N\} = d'_+ = \kappa\{y_{lb} + p_*y^N\}$ which implies that $\kappa\{y_{lb} + p_*y^N\}$ *is a sudden stop level of debt* if the pair (d_*, p_*) implies a sharp contraction in consumption and a reversal in the current account. The first fact was already shown. Note that the order of magnitude of this "recession" depends on the level of consumption reached in phase i) before the sudden stop which, given the unconstrained nature of this phase and the smoothness of consumption in that regime, is similar to c and bigger than c_*. Thus, *memory in the form of a sequence* $y^T_{T-n}, ..., y^T_{T-1}$, *matters in order to capture the quantitative properties of the sudden stop* and the GME is capable of capturing it. Technically, the compactness of the SCE insures that we can adjust the severity of the crises along with a finite lower bound on consumption, both proved in lemma 3
It remains to show that the jump from (d, p) to (d_*, p_*) generates a current account reversal. Let $d_+, d_{*,+}$ be the level of next period debt associated with c, c_* respectively. In order to generate the mentioned reversal, we must have $d_{*,+} - d_* < d_+ - d$. As $d_{*,+} = y_{lb} + p_*y^N < d_+ = y_{lb} + py^N$ and $d < d_*$, the desired result follows. In order to connect $d_{*,+} - d_*$ with the current account level before the sudden stop, note that, by the arguments used to show phase i), a sequence of negative shocks to income, $y_{t-n} = y_{lb}$ with $n = 2, 3...N$, will generate an increasing sequence of debt. Thus, $d_{t-n} > d_{*,+}$ without loss of generality. To sum up, the sudden stop generates a *reduction in consumption and a current account reversal* as desired. Finally, as this event happens for a particular trajectory of exogenous shocks $y_0, y_1, ..., y_T$ in finite time, the event has positive probability but can be considered a *"rare event"* as noted by Mendoza and Smith ((([50])).
It is sometimes observed that after a sharp depreciation, it follows a recovery in consumption coupled with a real appreciation and a current account improvement. This is the case of Portugal 2008 in phase iii). A GME is capable of replicating these facts as a path in the unconstrained regime. In particular, the model associates an increase in the national income with this type of phase iii). That is, we must observe an increase in $y^T_{T+1} - d_+$. This shock implies an increase in consumption which generates the real appreciation due to the intra-temporal optimality condition $p(c^T)$.[59] Moreover, the concavity of the utility function implies $d_{++} < d_+$, as it is possible to smooth consumption in an unconstrained economy. Thus, the observed current account improvement follows.

In order to show the existence of Fisherian deflation, we must prove that there exist a *binding* level of debt d_{++} which simultaneously satisfies equations (52) and (76) with $0 < d_{++} < \kappa\{y_{lb} + p_*y^N\} = d_{*,+}, c_+ < c_*$ and $p_+ < p_*$. That is, $d_{++} = \kappa\{y^T_+ + p_+y^N\}$ must satisfy:

$$p = \frac{A_2(y^N)}{A_1(y^T_+ + R^{-1}\kappa\{y_+ + p_+y^N\} - \kappa\{y_{lb} + p_*y^N\})} \tag{77}$$

By setting $y^T_+ > y_{lb}$, it is possible to choose $\kappa\{y^T_+ + p_+y^N\} < \kappa\{y_{lb} + p_*y^N\}$ and c_+ will remain positive. Note that the last 2 inequalities together imply $p_+ < p_*$. Moreover, p is increasing and real valued in c, which imply that we must have $c_+ < c_*$. Finally, to show optimality, d_{++} must satisfy:

$$\begin{aligned} &U'\{A_1(y^T_+ + R^{-1}\kappa\{y_+ + p_+y^N\} - \kappa\{y_{lb} + p_*y^N\})\} \geq \\ &\geq\ E(U'\{A_1(y^T_{++} - \kappa\{y_+ + p_+y^N\} + R^{-1}\overline{d})\}) \end{aligned}$$

[59]See Lemmas 20 and 21 in the appendix for section 4.

As $c_+ < c_*$, we know $U'\{A_1(c_+)\} > U'\{A_1(c_*)\}$. As $y_+ + p_+y^N < y_{lb} + p_*y^N$, it follows that $E(U'\{A_1(y^T_{++} - \kappa\{y_+ + p_+y^N\} + R^{-1}\overline{d})\}) < E(U'\{A_1(y^T_{++} - \kappa\{y_{lb} + p_*y^N\} + R^{-1}\overline{d})\})$, which implies that the condition above is satisfied. As mentioned before, the path of Spain in the 2008 sudden stop was characterized by a monotonically decreasing consumption sequence; even after the sharp drop in period "T". At the same time, we observed an improvement in the current account. The "Fisherian deflation" described above matches these facts as $d_{++} < d_+$, which implies the observed improvement in the current account, and $c_+ < c_*$, which gives us the decreasing consumption sequence.

Supplementary Material for section 5.3 (numerical procedure)

We now present the ergodic, stationary and non-stationary algorithm together with some details pf the procedure involved.

GME Ergodic Algorithm

Step 1: Computation

- Fix the vector of parameters from section 2.1, $[\kappa, \beta, \sigma, \xi, a] \equiv \Theta \gg \overrightarrow{0}$ with $U(c_1, c_2) = \frac{A(c_1,c_2)^{(1-\sigma)}}{1-\sigma}$ and $A(c_1, c_2) = (a(c_1)^{(1-1/\xi)} + (1-a)(c_2)^{(1-1/\xi)})^{\frac{1}{1-1/\xi}}$.
- Fix $Y \times K_1 \times K_2$
- Compute $d(d, y)$ from (21) *ignoring* the collateral constraint.
- Compute d_* from equation (72) in the appendix
- Compute $\varphi \in \Phi$ from equation (73) in the appendix

Step 2.1: Stationary simulation

- Take a "draw" of length $T+1$ from (Y, q), the exogenous Markov process which generates tradable output.
- Fix $[d_{T-1}, y_{T-1}]$ from $Y \times K_2$, obtain d_T from $d(d, y)$. Verify if the collateral constraint binds and adjust d_T if necessary. Then compute $d_{T+1}(y_T, d_{T-1}, y_{T-1})$ from $\varphi \in \Phi$ for every $y_T \in Y$. Compute the rest of the endogenous variables from equations (52)-(56).
- Take $[p_T(y_T), d_T(y_{T-1})]$ as given from the previous • and compute $[p_{T-1}(y_{T-1}), d_{T-1}(y_{T-2})]$ from equations (52)-(56). Note that d_{T-1} is in the preimage of $d'(d, y)$ with $d_T = d(d_{T-1}, y_{T-1})$.
- Repeat the above • until you get $[p_0(y_0), d_0]$, where d_0 is allowed to be independent of y_0 as they are both initial conditions of the system.

Step 2.2: Ergodic simulation

- Take a "draw" of length $N+1$ from (Y, q), the exogenous markov process which generates tradable output.
- Compute $[p_j(y_j), d_j(y_{j-1})]$ for $j = N+1, N, N-1$ as in the stationary simulation procedure
  - If d_N binds, $y_{N-1} - d_{N-1} + d_N/R > y_{LB} - d_* + d(d_*, y_{LB})/R$ and $d_N > d(d_*, y_{LB})$, the process hits the atom and reverts to it.
- Continue until $[p_0(y_0), d_0]$

The numerical procedure is based on the policy function of the RE, $d'(.,.)$. It is computed as a solution to (21) without a collateral constraint. Then the atom d_* is computed using $d'(.,.)$ in (72). Note that if the process hits the constraint in period t, then $d(d_t, y_t) > d(d_*, y_{lb})$. Thus in order to find a regeneration point, we need:

$$U'\left\{c_1\left(y_t - d_t + d(d_t, y_t)/R\right)\right\} - \beta R \sum_{y'} U'\left\{c_1\left(y' - d'(d_t, y_t) + d''(d_t, y_t, d', y')\right)\right\} \equiv f(d''(y'); d, d', y_t) \geq 0$$

$$U'\left\{c_1\left(y_{lb} - d_* + d'(d_*, y_{lb})/R\right)\right\} - \beta R \sum_{y'} U'\left\{c_1\left(y' - d'(d_*, y_{lb}) + d''(d_*, y_{lb}, d', y')\right)\right\} \geq 0$$

where, for exposition purposes, we denote $d_+ = d'$ and $d_{++} = d''$ if the collateral constraint does not bind or if it binds with equality. Note that in section 4 we require d'' to be independent of d, d', y. This fact implies that we can easily find d'' but there are some additional conditions which must be satisfied by paths in order to revert to the atom (i.e. consumption and debt must be greater than the atomic level). This restrictions affects the frequency at which the atom is hit. Thus, in order to improve the recurrent structure of sets and gain computationally efficiency, we modified the selection from the GME. When we allow d'' to depend on an expanded state space, we can find a stationary Euler equation for each d, y as d' has at most 2 solutions when the collateral constraint is binding.
To formally prove the existence of this selection is beyond the scope of this paper. We need to show that $f(d''(y'); d, d', y_t) \geq 0$ has a minimum in d'' for each $y'; d, d', y_t$, subject to the collateral constraint, and that the value function of this problem is equal to zero if the collateral constraint is not binding. Numerically, this selection is easily implemented and the algorithm is really fast as we do not need to compute every selection of the GME.

GME Non-Stationary Algorithm

Step 1

- Fix the vector of parameters from section 2.1, $[\kappa, \beta, \sigma, \xi, a] \equiv \Theta \gg \overrightarrow{0}$
- Fix $Y \times K_1 \times K_2$ in order to define a compact set for (y, p, d) respectively.
- Fix a $[p_{T+2}(y_{T+2}), d_{T+2}(y_{T+1})]$ for each $(y_{T+1}, y_{T+2}) \in Y \times Y$. This will define a selection $\varphi \in \Phi$ using the Euler equation in time T.

Step 2

- Take a "draw" of length $T+1$ from (Y, q), the exogenous markov process which generates tradable output.
- Fix $[p_{T+1}(y_{T+1}), d_{T+1}(y_T)]$ from $K_1 \times K_2$ and compute $[p_T(y_T), d_T(y_{T-1})]$ from equations (52)-(56)
- Take $[p_T(y_T), d_T(y_{T-1})]$ as given from the previous • and compute $[p_{T-1}(y_{T-1}), d_{T-1}(y_{T-2})]$ from equations (52)-(56). Include $[p_{T+1}(y_{T+1}), d_{T+1}(y_T)]$ in the Euler equation. This step implies a change in the selection from Φ and, thus, breaks the stationarity of the process.
- Repeat the above • until you get $[p_0(y_0), d_0]$, where d_0 is allowed to be independent of y_0 as they are both initial conditions of the system.