Time Consistent Policies and Quasi-Hyperbolic Discounting

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Summary

In dynamic choice models, dynamic inconsistency of preferences is a situation in which a decision-maker's preferences change over time. Optimal plans under such preferences are time inconsistent if a decision maker has no incentive to follow in the future the (previously chosen) optimal plan. A typical exemplification of dynamic inconsistency is so called "present bias", i.e. the repetitive preference towards smaller present rewards versus larger future rewards.

The study of dynamic choice of decision makers who possess dynamically inconsistent preferences has long been the focal point of much work in behavioral economics. Experimental and empirical literatures both point to the importance of various forms of present-bias. The canonical model of dynamically inconsistent preferences exhibiting present-bias is a model of quasi-hyperbolic discounting. A quasi-hyperbolic discounting model is a dynamic choice model, in which the standard exponential discounting is modified by adding an impatience parameter that additionally discounts the immediately succeeding period.

A central problem with analytical study of decision makers who possess dynamically inconsistent preferences is how to model their choices in sequential decision problems? One general answer to this question is to characterize and compute (if they exist) constrained optimal plans, that is optimal among time consistent sequential plans. Time consistent plans are those among the set of feasible plans that will be actually followed, or not re-optimized, by agents whose preferences change over time. These are called Time Consistent Policies (TCPs).

This chapter presents many results on the existence, uniqueness, and characterization of stationary, or time invariant, TCPs in a class of consumption-savings problems with quasi-hyperbolic discounting, as well as provides some discussion of how to compute TCPs in some extensions of the model. In this chapter, the role of the *generalized Bellman equation operator* approach is central. This approach allows to provide sufficient conditions for the existence of time consistent solutions and facilitates their computation.

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Importantly, the generalized Bellman approach can also be related to a common first order approach in the literature known as the *generalized Euler equation* approach. By constructing sufficient conditions for continuously differentiable TCPs on the primitives of the model, sufficient conditions under which a generalized Euler equation approach is valid can be provided.

Other important questions addressed in the chapter include sufficient conditions for the existence of monotone comparative statics in interesting parameters of the decision environment, as well as generalizations of the generalized Bellman approach to allow for unbounded returns and general certainty equivalents. The case of multi-dimensional state space is also discussed, as is a general self-generation method for characterizing non-stationary TCPs.
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INTRODUCTION

The relationship between discounting and dynamic choice has been a topic of great interest in economics since the early paper of Ramsey (1928) almost a century ago. In his work, Ramsey proposed to model intertemporal preferences of a decision-maker as the weighted sum of future utilities. Subsequently, Samuelson (1937) sharpened this model of intertemporal preferences by proposing a model with exponential discounting, and this model soon became a standard approach to specifying intertemporal preferences in dynamic economies. Then, with the subsequent work of Koopmans (1960) on the axiomatic approach to recursive utility (with exponential discounting as a special case), the foundations for dynamically consistent preferences was made foundational.[1]

The seminal paper of Strotz (1956) challenged this orthodoxy. Strotz proposed a model of dynamic choice for decision makers with dynamically *inconsistent* preferences. Since his pioneering suggestion, there has been a large and ever growing literature in dynamic behavioral economics which has sought to understand dynamic decisions under such preferences. This work has appeared in such diverse fields in economics as macroeconomics, financial economics, political economy, environmental economics, health economics, and public finance. The need for such studies has been motivated by a large and growing empirical and experimental literature that has shown how important preference reversals is in the understanding of dynamic choice of decision-makers that are forced to compare current vs. future utilities.[2]

[1] See also Halevy (2015) for a nice recent discussion of these issues.

[2] For work discussing the empirical motivation for the importance of present-bias and dynamic incon-

There has been at least two strands of work on the theoretical side of this literature. One strand is found in a series of papers which axiomatize preferences that exhibit various forms of dynamic inconsistency related to discounting. This includes the work of Montiel Olea and Strzalecki (2014), Chambers and Echenique (2018), and Chakraborty (2021) on quasi-hyperbolic discounting and related forms of present bias; Galperti and Strulovici (2017) on dynastic preferences with altruism and their relationship with quasi-hyperbolic discounting; Jackson and Yariv (2015) and Lizzeri and Yariv (2017) concerning the study of dynamic collective choice and time inconsistency; temptation preferences axiomatized in Gul and Pesendorfer (2001, 2004, 2005); and other related work on self-control presented in Noor (2011) and Dekel and Lipman (2012), for example. This line of work has also looked at the axiomatization of "naive" vs. "sophisticated" decisions for dynamically inconsistent consumers (e.g., the papers of Ahn et al. (2019), and Ahn et al. (2020)) in the context of quasi-hyperbolic preferences.

The second strand includes voluminous literature which seeks to characterize time consistent choice in models where agents have dynamically inconsistent preferences. Starting with the early work of Phelps and Pollak (1968), Pollak (1968) and Peleg and Yaari (1973) where the so-called "game theoretic" interpretation of time consistent choice was proposed, this work has sought to identify sufficient conditions for the existence of Stationary Markov equilibria in models of quasi-hyperbolic discounting (e.g., see the work of Laibson (1997), Harris and Laibson (2001), Krusell and Smith (2003), Krusell et al. (2010), Nowak (2010), Harris and Laibson (2013), Chatterjee and Eyigungor (2016), Balbus et al. (2015c, 2018), Cao and Werning (2018), Jaśkiewicz and Nowak (2021), and Bäuerle et al. (2021), among others), as well as work that studies the long-memory subgame perfect equilibria as in the papers of Bernheim et al. (2015) or Balbus and Woźny (2016). More recently, these results have been extended to more general behavioral discounting cases as in the papers of Balbus et al. (2021, 2022), Jensen (2021), and Richter (2020).

In this literature, the question of defining what a "time consistent solution" actually is turns out to be a very important question. Since the work of Phelps and Pollak (1968) and Peleg and Yaari (1973), as well as much of the literature that has followed, researchers have viewed a "time consistent solution" for an individual consumer as a

sistency in choice, see Angeletos et al. (2001), Ameriks et al. (2007), Halevy (2015), and Cohen et al. (2020).

subgame perfect equilibrium of a dynamic intrapersonal game between a collection of that consumer's "selves". That is, it envisions the consumer as playing a dynamic game between one's current self and each of her future "selves" or "generations", with the appropriate equilibrium concept defining a time consistent solution as a subgame-perfect Nash equilibrium (SPNE, henceforth).

It is important to note that this definition differs from the solution concept proposed by Strotz (1956), and further studied by Pollak (1968) and Kydland and Prescott (1977), where a decision theoretic viewpoint was taken, and the idea of *optimal* time consistent policies was proposed where one views the time consistent solution as being "optimal" relative to the set of all possible time consistent solutions.[3] In some cases, the two approaches can be linked, but there is no *equivalence* between the two notions. As Caplin and Leahy (2006), Balbus et al. (2015c), and Balbus et al. (2022) discuss in some detail, although the optimal time consistent plan is a SPNE of some game, the converse is false as from the vantage point a decision maker relative to how future ties are being broken (in favour of a current self vs. future selves). It also bears mentioning, the set of SPNE may also be large, and most importantly, not necessarily possess the element with the *greatest value*. Hence, an *optimal* SPNE (i.e., a SPNE that corresponds to some optimal time consistent policy) may simply not exist.

This chapter studies the question of time consistent solutions from the game theoretic approach in dynamic models where agents have present bias. The remainder of the chapter is layout as follows. Section 2 (*The quasi-hyperbolic discounting model*) describes a prototype of the quasi-hyperbolic discounting model, and provide a detailed discussion of its structure starting with the finite horizon case. Here, an example compares naive vs. sophisticated solution, as well as describes what is commonly referred to as a "generalized Euler equation" in the existing literature (e.g., see Harris and Laibson (2001) and Krusell and Smith (2003)). Section 3 (*Searching and characterizing stationary Markov perfect equilibrium*) considers the infinite horizon case for a version of the quasi-hyperbolic model following a (stochastic) game-theoretic interpretation of time consistent plans (TCPs), and define a time consistent solution to be stationary Markov perfect equilibrium (SMPE) in the resulting dynamic intrapersonal game. The new idea of a "Generalized Bellman equation" for the quasi-hyperbolic consumer is introduced, and this functional equation plays a critical role in the construction of the SMPE. We

[3] See Caplin and Leahy (2006) for a nice discussion of the idea of optimal time consistent policies.

also give sufficient conditions for the existence, characterization, and computation of time consistent equilibrium as SMPE, and as well as discuss when it is unique. Examples of applications of the results are provided, as well as a set of monotone comparative statics results relative to the set of time consistent plans. The question of sufficient conditions for the existence of a generalized Euler equation governing the model's SMPE for the infinite horizon case is also addressed. In section 4 (*Extensions*) of the paper, extensions of the model to more general settings are considered including unbounded state spaces and time inconsistent recursive utilities. In this section, the idea of generalized certainty equivalents is discussed, as well as the question of how to extend the results to models with multidimensional states. Finally, in section 5 (*Self-generation approach*) of the chapter, self-generation approaches are provided that can be used to construct more general notions of time consistent plans as subgame perfect equilibria.

THE QUASI-HYPERBOLIC DISCOUNTING MODEL

A PROTOTYPE MODEL

The canonical version of the consumption and savings problem for a consumer with quasi-hyperbolic preferences motivates much of the discussion in this chapter. Consider a discrete time, T-period, consumption-saving model, where the sequence of lifetime preferences, index by date t, over sequences of consumption $(c_{t+\tau})_{\tau=0}^{T-t}$ is given by:

$$u(c_t) + \mathbb{E}_t \sum_{\tau=1}^{T-t} \beta\delta^\tau u(c_{t+\tau}). \tag{1}$$

Here $\delta \in (0,1)$ is called a long-term discount factor while $\beta \in (0,1]$ is an additional short-term impatience parameter, $\mathbb{E}_t$ denotes the mathematical expectation taken relative to date t information, and T for the moment is finite. Preferences given by (1) are the same for any t and, hence, are time invariant. For given t these preferences are non-stationary whenever $\beta < 1$. Indeed, the ratio of discounted utilities in any two time periods $t+\tau+1$ and $t+\tau$ (for $\tau > 0$) is given by:

$$\frac{\beta\delta^{\tau+1}u(c_{t+\tau+1})}{\beta\delta^\tau u(c_{t+\tau})},$$

and hence is equal to $\frac{\delta u(c_{t+\tau+1})}{u(c_{t+\tau})}$ for $\tau > 0$ as compared to $\frac{\beta\delta u(c_{t+1})}{u(c_t)}$ for $\tau = 0$. These two properties, i.e. time invariance and non-stationarity, imply time inconsistency of sequences of preferences given by (1). Indeed, a decision maker planning choices over consumptions τ periods ahead would have an incentive to change his choices, when this period actually occurs. See Halevy (2015) proposition 4 for a formal argument. For $\beta < 1$, preferences exhibit present bias: i.e., a decision maker who planned a choice of $(c_{t+\tau})_{\tau=0}^{T-t}$ at time t will have an incentive to increase his consumption at the cost of reducing investment when period $t' = t+\tau$ actually occurs (as $\beta\delta < \delta$). In what follows, the consumer thinks of herself as a sequence of successive "selves", each self associated with particular time period t.

In any date t, self t enters the period with a stock $s_t \in S$ (where $S = [0, \bar{s}]$) that is divided between the current consumption $c_t \in [0, s_t]$) and current savings/investment denoted by $i_t := s_t - c_t$, where savings/investment is placed into a (possibly stochastic) return technology. The "stock" s_t is interpreted as an productive asset or a capital. The current investment parameterizes transition probability over tomorrow's stock S and is given by the first order Markov process $Q(\cdot|i_t)$. This specification is very general and allows to cover many special cases. For example, it allows for $s_{t+1} = F(\omega_{t+1}, i_t)$ where $\omega_{t+1} \in \Omega$ is a random shock with distribution π_{t+1} on Ω and F a (production or transformation) function. Here F can be multiplicative with $F(\omega_{t+1}, i_t) = \omega_{t+1} g(i_t)$ (in which case ω_{t+1} can be interpreted as a random productivity shock) or additive $F(\omega_{t+1}, i_t) = g(i_t) + \omega_{t+1}$, (where ω_{t+1} can be interpreted as a random labour income) for some continuous and increasing g. One can rewrite this transition process as: $Q(A|i_t) = \int_\Omega \mathbf{1}_A(F(\omega, i_t))\pi_{t+1}(d\omega)$, where $\mathbf{1}_A()$ is a indicator function of a Borel set $A \subseteq S$. Finally, this specification also allows for a deterministic state transitions. Indeed, consider a typical transition between current and next period capital: $k_{t+1} = F(k_t) - c_t$ for some increasing and continuous production function $F : \mathbb{R}_+ \to \mathbb{R}_+$ and full depreciation. Introducing $s_t := F(k_t)$ as a state variable and $i_t := F(k_t) - c_t$ one obtains the transition given by $Q(A|i) = \mathbf{1}_A(F(i))$.

THREE PERIOD EXAMPLE

It is useful to first consider a simple example of the model with $T = 3$ to understand the complications that quasi-hyperbolic discounting creates. The preferences of the date 1 self 1 are defined then over (c_1, c_2, c_3), and given by

$$u(c_1) + \beta\delta(u(c_2) + \delta u(c_3)),$$

while the preferences of date 2 self are defined over (c_2, c_3) and given by:

$$u(c_2) + \beta\delta u(c_3)$$

with $u(c_3)$ being the preferences for the date 3 self. In what follows, assume the utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave and twice continuously differentiable. Additionally impose the Inada condition on u' so that consumption choices are interior. To keep things illustrative, consider the special case of deterministic linear technology $F(k)$, and associate state s_t with returned capital $(1+r)k_t$. That is $s_t := F(k_t) = (1+r)k_t$ for some given interest $r > 0$.

Since the model encompasses multiple selves, its solution depends on the beliefs that the current self formulates about all of his future selves in any given period. The literature consider two standard cases of such a belief. The first is the "naive" solution, in which the current self (mistakenly) believes that her future selves will continue the proposed plan. The second belief is the "sophisticated" solution, in which each current self correctly forecasts changes to the proposed plan by any future self.

For $T = 3$, the naive solution is first characterized, followed by the sophisticated solution and a discussion on how they can differ. Consider a simple model in which self 1 is endowed with a state (income or production) s_1 and divides it between c_1 and investment i_1. Such investment increased by the interest rate is returned to self 2. Consumption is nonnegative but otherwise assets can be freely moved between the periods.[4]

The naive solution to quasi-hyperbolic model To solve for the naive solution, start with date $t = 1$ self. The agent solves for (c_1, c_2, c_3) first:

$$\max_{(c_i \geq 0)_{i=1}^{3}} u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3)$$
$$\text{s.t.} c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} \leq s_1,$$

Observe that only c_1 is actually realized from the optimal plan (c_1, c_2, c_3). That is, because preferences are dynamically inconsistent, the continuation plan (c_2, c_3) is then

[4] Although not covered in this survey, credit/borrowing constraint are important to understand dynamics of solutions to models without commitment. This is especially important for models with commitment assets. See e.g. Laibson (1997) and Woźny (2016).

updated by date $t = 2$ self by solving

$$\max_{(c_i \geq 0)_{i=2}^{3}} u(c_2) + \beta\delta u(c_3)$$
$$\text{s.t.} c_2 + \frac{c_3}{(1+r)} \leq (1+r)(s_1 - c_1) := s_2.$$

Finally, since period $t = 3$ self makes no choice (i.e. $c_3 = s_3$) the actual path is then c_1 chosen by date 1 self and (c_2, c_3) chosen by self 2.

The sophisticated solution to quasi-hyperbolic model Strotz (1956) proposed a sophisticated solution to the model with changing preferences (i.e. a solution that will "actually be followed") To find such a solution, solve the problem by backward induction starting from date $t = 2$:

$$\max_{c_2 \geq 0, c_3 \geq 0} u(c_2) + \beta\delta u(c_3)$$
$$\text{s.t.} c_2 + \frac{c_3}{(1+r)} \leq (1+r)k_2 := (1+r)(s_1 - c_1).$$

Here, assume self 2 is endowed with $(1+r)k_2 = s_2$. The problem is strictly concave and hence the solution unique. The first order condition of the optimal interior allocation is as follows:

$$\frac{u'(c_2)}{\beta\delta u'(c_3)} = 1 + r.$$

or

$$\frac{u'(c_2)}{\beta\delta u'((1+r)(k_2(1+r) - c_2)} = 1 + r.$$

where everything here is standard.

Denote the optimal choice $c_2^*(k_2)$ and $c_3^*(k_2)$. These are now called the *reaction curves* of the second period self as they depend on the assets / capital obtained from the first period self. To simplify, denote by $k_3^* := k_2(1+r) - c_2^*(k_2)$. Clearly: $c_2^*(k_2) = k_2(1+r) - k_3^*(k_2)$ and $c_3^*(k_2) = (1+r)k_3^*(k_2)$. Using the implicit function theorem, the interior optimal reactions c_2^*, c_3^* and k_3^* are continuously differentiable functions in the asset/capital.

Now move to consider period 1 self problem:

$$\max_{c_1 \geq 0, k_2 \geq 0} u(c_1) + \beta\delta u(c_2^*(k_2)) + \beta\delta^2 u(c_3^*(k_2))$$
$$\text{s.t.} c_1 + k_2 \leq s_1.$$

That it, the first period self correctly forecasts self 2 (and 3) reaction to the left asset/capital k_2. The problem is continuous hence the solution exists. Substituting, one obtains:

$$\max_{k_2 \leq s_1} u(s_1 - k_2) + \beta\delta u(k_2(1+r) - k_3^*(k_2)) + \beta\delta^2 u((1+r)k_3^*(k_2)).$$

As the reaction function k_3^* is differentiable, one obtains the first order condition for optimal interior k_2:

$$-u'(c_1^*) + \beta\delta u'(c_2^*(k_2))[1 + r - \frac{\partial k_3^*(k_2)}{\partial k_2}] + \beta\delta^2(1+r)u'(c_3^*(k_2))\frac{\partial k_3^*(k_2)}{\partial k_2} = 0. \tag{2}$$

Recall, for self 2, $u'(c_2^*(k_2)) = \beta\delta(1+r)u'(c_3^*(k_2))$ for any k_2. Using this

$$-u'(c_1^*) + \beta\delta u'(c_2^*(k_2))[1 + r - \frac{\partial k_3^*(k_2)}{\partial k_2}] + \delta u'(c_2^*(k_2))\frac{\partial k_3^*(k_2)}{\partial k_2} = 0,$$

and hence:

$$\frac{u'(c_1^*)}{\beta\delta u'(c_2^*(k_2))} = [1 + r - \frac{\partial k_3^*(k_2)}{\partial k_2}] + \frac{1}{\beta}\frac{\partial k_3^*(k_2)}{\partial k_2}$$

or:

$$\frac{u'(c_1^*)}{\beta\delta u'(c_2^*(k_2))} = 1 + r + [\frac{1}{\beta} - 1]\frac{\partial k_3^*(k_2)}{\partial k_2}.$$

Krusell et al. (2002) called this equation a "generalized Euler equation". Indeed, it has the standard interpretation of the marginal rate of substitution being equal to the ratio of prices, but with the critical addition of the *corrective* factor $[\frac{1}{\beta} - 1]\frac{\partial k_3^*(k_2)}{\partial k_2}$ that involves the policy function itself. Notice, this term disappears when $\beta = 1$ (the dynamically consistent preference case). Recall, the term $\frac{\partial k_3^*(k_2)}{\partial k_2}$ can be computed using the implicit function theorem:

$$\frac{\partial k_3^*}{\partial k_2}(k_2) = \frac{(1+r)\beta\delta u''(c_2^*(k_2))}{u''(c_2^*(k_2)) + (1+r)\beta\delta u''(c_3^*(k_2))}.$$

If $\beta \neq 1$, then the generalized Euler equation implies, among other things, that in order to characterize the optimal investment k_2, the first period self must *use* the (investment) reaction curve of the second period self in defining her optimal solution. This is a critical feature of this model, and is the case as from the perspective of the first period self some part of its investment will be misused for the excessive consumption in period 2. The fact that this *corrective* factor disappears for $\beta = 1$ results exactly from the principle of optimality. Indeed, whenever the preferences are *consistent*, by the principle of optimality (and envelope theorem), the reaction curve of the future self is still optimal from the perspective of the current self.

NOTATION AND MORE THAN THREE PERIODS

For future reference observe that the problem of the first period self can be simplified by the introduction of the continuation value. This is, the value from consuming second period self choices c_2^* and c_3^* but evaluated from the perspective of the first period self:

$$V_2(k_2) := u(c_2^*(k_2)) + \delta u(c_3^*(k_2)) = u((1+r)k_2 - k_3^*(k_2)) + \delta u((1+r)k_3^*(k_2))$$

Then, the problem of the first period self can be simplified to:

$$W(k_1) := \max_{k_2 \leq (1+r)k_1} u((1+r)k_1 - k_2) + \beta\delta V_2(k_2),$$

where the value to the fist period self problem is denoted by $W(k_1)$. V_2 is differentiable as c_2^* and c_3^* are and hence the first order condition for the optimal choice of k_2^* is simply given by:

$$u'((1+r)k_1 - k_2^*) = \beta\delta V_2'(k_2^*).$$

By the "envelope":

$$V_2'(k_2) = (1+r)u'(c_2^*(k_2)) - \frac{\partial k_3^*}{\partial k_2}(k_2)[u'(c_2^*(k_2)) - (1+r)\delta u'(c_3^*(k_2))],$$

one obtains the same first order condition as in (2).

It turns out, unfortunately, that this solution approach suggested above of composing the reaction curves and constructing a first order characterization of consistent plans for the sophisticated consumer who lives for $T = 3$ periods cannot be easily extended to more periods. To see this is the case, consider her problem for the $T = 4$ model. Repeating the above reasoning, c_3^*, c_4^* and k_4^* are unique and continuously differentiable. The objective of the second period self is continuous and differentiable. It may happen, however, that now the argmax for the second period self k_2^* is *not* unique (e.g., when strict concavity of the objective fails). Moreover, any selection from the argmax certainly need not be continuously differentiable, and might actually be discontinuous. Hence, the problem of the period 1 self may not be well-defined (i.e., might not possess an optimal solution for some initial capital levels). Notice also, the generalized Euler equation representation of sequential solutions no longer holds.

Finally, let us also mention a special case for which the above mentioned problems are not relevant and the generalized Euler equation approach is useful, and naive and sophisticated solutions coincide. This example is unfortunately a "knife-edge" but does admit a closed-form solutions for time consistent plans.

Example 1 (The naive and sophisticated solutions coincide for the log utility) *Take* $u(c) = \ln(c)$. *The naive solution to the problem:*

$$\max_{\{c_i \geq 0\}_{i=1}^3} \ln(c_1) + \beta\delta \ln(c_2) + \beta\delta^2 \ln(c_3)$$
$$s.t. c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} \leq s_1,$$

gives:

$$c_1 = \frac{s_1}{1+\beta\delta+\beta\delta^2} \quad k_2 = \frac{(\beta\delta+\beta\delta^2)s_1}{1+\beta\delta+\beta\delta^2}.$$

Sophisticated solutions produces reactions:

$$c_2^* = \frac{(1+r)k_2}{1+\beta\delta} \quad c_3^* = \frac{\beta\delta(1+r)^2 k_2}{1+\beta\delta}$$

and the 1st period self preferences:

$$\ln(c_1) + \beta\delta \ln(c_2^*(k_2)) + \beta\delta^2 \ln(c_3^*(k_2)) = \ln(c_1) + \beta\delta \ln(k_2) + \beta\delta^2 \ln(k_2) + constant.$$

This constant term does not depend on k_2. *This gives the same choice:*

$$c_1^* = \frac{s_1}{1+\beta\delta+\beta\delta^2} \quad k_2^* = \frac{(\beta\delta+\beta\delta^2)s_1}{1+\beta\delta+\beta\delta^2}$$

As the first period choice of capital k_2^* *is the same, hence the choice of* c_2 *and* c_3 *also coincide for the naive and sophisticated case.*

The log utility case is a special one as the reactions are linear. The properties of the log utility case allow one to extend the generalized Euler equation approach to more periods, actually including $T = \infty$.

We show how this general class of problems for the stochastic state transition can be addressed using the so called generalized Bellman equation approach.

INFINITE HORIZON AND STATIONARY MARKOV PERFECT EQUILIBRIA

Consider now the infinite time horizon ($T = \infty$) and a general model with preferences as given in (1). Again, assume the state space S is a bounded interval with $S = [0, \bar{s}]$. The initial states is s_1. Suppose that the current period t is fixed and the consumer owns $s \in S$ amount of capital or stock. The consumer selects $c \in [0, s]$ as a consumption and leaves $i = s - c$ as an investment. The budget constraint is hence $c + i = s$. The next period states s' is controlled by the current investment and (generally) the current capital stock s via the transition distribution: $s' \sim Q(\cdot|i, s)$.

Generally, a feasibly consumption policy in period t can be any measurable function of time as well as history of states up to the current period, i.e. $(s_\tau)_{\tau=1}^t$. A consumption policy is Markov if it depends on time t and the current state only (i.e. s_t). In addition, it is Markov stationary if it is time invariant and depends on s_t only. It is formalized in the next definition.

Definition 1 (Stationary Markov policy) *The stationary Markov policy (SMP) is a Borel measurable function $h : S \mapsto S$ such that $h(s) \in [0, s]$ for any $s \in S$.*

The set of all SMP is defined as follows:

$$\mathcal{H} := \{h : S \mapsto S : h \text{ is Borel measurable, and } h(s) \in [0, s] \text{ for all } s \in S\}. \tag{3}$$

Assume, consumer in period t predicts the future consumption is consistent with a SMP $h \in \mathcal{H}$. That is, $h(s_\tau)$ is consumed in the period $t + \tau$ whenever the capital in this period is s_τ. The evolution of the capital stocks from $t + 1$ onward is hence a Markov chain $(s_\tau)_{\tau=1}^\infty$ whose transition probability from s_τ to $s_{\tau+1}$ takes a general form $Q(\cdot|s_\tau - h(s_\tau), s_\tau)$.

To evaluate the utility from an aggregated consumption of future selves one uses the continuation value function.

Definition 2 (Continuation Value) *The continuation value function (CV) of future selves following the stationary policy $h \in \mathcal{H}$ is defined as follows*

$$V_h(s') := \mathbb{E}\left(\sum_{\tau=1}^{\infty} u(h(s_\tau))\delta^{\tau-1}|s_1 = s'\right).$$

The expected utility of the consumer today is hence:

$$P(c, s; h) = \mathbb{E}\left(u(c) + \beta\delta\left(\sum_{\tau=1}^{\infty} u(h(s_\tau))\delta^{\tau-1}\right)\right).$$

By definition of V_h and the property of the conditional expectation operator, letting $s_1 = s$ one obtains:

$$\begin{aligned} P(c, s; h) &= u(c) + \beta\delta\mathbb{E}\left(\sum_{\tau=1}^{\infty} u(h(s_\tau))\delta^{\tau-1}\,|s_1\right) \\ &= u(c) + \beta\delta \int_S V_h(s')Q(ds'|s - c, s). \end{aligned}$$

Next, introduce the correspondence returning the best policies for the current self against the stationary policy of future selves.

Definition 3 (Best Response Correspondence) *For any $h \in \mathcal{H}$, define the Best Response correspondence (BR) as follows*

$$BR(h) := \{h' \in \mathcal{H} : h'(s) \in arg \max_{i \in [0,s]} P(c, s; h) \text{ for any } s \in S\}.$$

Define a Stationary Markov Perfect Equilibrium as follows.

Definition 4 (Stationary Markov Perfect Equilibrium) *A policy $h^* \in \mathcal{H}$ is a Stationary Markov Perfect Equilibrium (SMPE) if for any $s \in S$,*

$$P(h^*(s), s; h^*) \geq P(c, s; h^*).$$

In other words, SMPE is the consumption plan h^* for every self that is optimal provided all the future selves stick to the same plan, namely h^*. Mathematically h^* is SMPE if and only if it is a fixed point of the best response correspondence BR defined on $\mathcal{H}$. Plan $h^* \in \mathcal{H}$ which is a SMPE remains an equilibrium if players (selfs) are allowed to use more general history dependent strategies. Terms Time Consistent Policy (TCP) and SMPE are used interchangeably.

GENERALIZED BELLMAN EQUATION

In the case of $\beta = 1$, the model reduces to a standard exponential discounting, and dynamic programming techniques can be applied to the consumers dynamic choice problem as the principle of optimality holds. When $\beta \in (0, 1)$, the consumer preferences exhibit present-bias, and her preferences are dynamically inconsistent as in the motivating example, and a new approach to computing TCP solutions has to be proposed. In this section, for the quasi-hyperbolic consumer, a "generalized" Bellman equation can be defined that provides a recursive formulation of non-recursive dynamic optimization problem. This approach helps to provide a characterization of how the quasi-hyperbolic discounter differs from a standard exponential discounter.

To develop a generalized Bellman equation for this problem, notice first the continuation value function in any SMP $h \in H$ must solve the following functional equation:

$$V_h(s) = u(h(s)) + \delta \int_S V_h(s')Q(ds'|s - h(s), s). \tag{4}$$

Moreover, the same h must solve the current self optimization problem:

$$W_h(s) := \max_{c \in [0,s]} u(c) + \beta\delta \int_S V_h(s')Q(ds'|s-c,s). \quad (5)$$

Then, the following is immediately implied:

$$V_h(s) = \frac{1}{\beta} W_h(s) - \frac{1-\beta}{\beta} u(h(s)). \quad (6)$$

Equation (6) is called the *generalized Bellman equation.*

Whenever h is a SMPE and V_h a corresponding continuation value, (h, V_h) solves the generalized Bellman equation. Similarly, whenever a pair (h, V_h) solves the generalized Bellman equation, with h being some measurable argmax selection from the maximization problem[5] then h is SMPE and V_h its corresponding continuation value.

Equation (6) is also a generalized Bellman equation as the element $\frac{1-\beta}{\beta} u(h(s))$ is the adjustment factor that must be made to the standard Bellman operator to account for changing preferences. Clearly, for $\beta = 1$, equation (6) reduces to the standard Bellman equation.

Based on equation (6), one defines an operator whose fixed points, say v^*, correspond to values for some pure strategy Markovian equilibrium policy. From there, one can recover the set of (pure strategy) Markovian equilibrium policy functions. Indeed, by $\mathcal{V}$ denote the set of feasible continuation value functions. Then for any $v \in \mathcal{V}$ define:

$$A(v)(s) = \max_{c \in [0,s]} \left\{ u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s) \right\},$$

and

$$B(v)(s) = \arg\max_{c \in [0,s]} \left\{ u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s) \right\}.$$

If $B(v)$ is single valued, then implementing (6) one constructs the generalized Bellman operator as follows:

$$T(v)(s) = \frac{1}{\beta} A(v)(s) - \frac{1-\beta}{\beta} u(B(v)(s)).$$

Observe that if $v^* \in \mathcal{V}$ is a fixed point of T, then it is a continuation value function supported by a SMPE; that is $v^* = V_{h^*}$ where h^* is a SMPE. Moreover, having v^* one computes $h^*(s) = B(v^*)(s)$. If $B(v)$ is not single valued the generalized Bellman equation can still be used to construct v^* by taking some selection from the argmax correspondence.

[5] That is: $W_h(s) = u(h(s)) + \beta\delta \int_S V_h(s')Q(ds'|s-h(s),s)$ for any s. .

SEARCHING AND CHARACTERIZING STATIONARY MARKOV PERFECT EQUILIBRIUM

The problem of existence of Markov Perfect Equilibrium is generally an open question. Even if one asserts its existence, not much is known about the characteristics or structure of the equilibrium set. An issue of especial importance is also uniqueness of Markov Perfect equilibrium. For that reason, an effective algorithm for computing of equilibria is hard to construct. For example, the successive iterations of the best response map need not be convergent, since the best response map is mostly neither monotone nor is a contraction mapping.

Assumptions that guarantee existence of equilibrium are first provided; then, under more restrictive assumptions, uniqueness of equilibrium is resolved (as well as convergence of successive iterations). One can search the equilibria in two alternative ways:

- **searching through policies**: find $h^* \in \mathcal{H}$, that is the best against itself. In other words, find h^* that is a fixed point of the best response map, i.e. $h^* \in BR(h^*)$.
- **searching through continuation value functions**: find $v^* \in \mathcal{V}$ that is a fixed point of generalized Bellman operator and find the policy h^* whose continuation value function is v^*. In other words, $v^* = T(v^*)$, and one reconstructs SMPE by the formula $h^* = B(v^*)$.

Both approaches are equivalent but sometimes one is more useful than another depending on the application at hand. In the remaining subsections, we use searching through policies to verify existence of equilibrium, while searching through continuation value functions to show its uniqueness.

EXISTENCE OF SMPE. SEARCHING THROUGH POLICIES

This subsection is focused on the problem of existence of SMPE which is a fixed point of the correspondence BR. Before stating the main theorem, let us provide an example that constructs an analytical solution for logarithmic preferences.

Example 2 *Let $S = (0, \infty)$, $u(c) = \ln(c)$ and suppose*

$$Q(S_0|i) = \pi(\{\omega \in \Omega : \omega i^\alpha \in S_0\}),$$

$c, i \in [0, s]$ and $i + c = s$, $s > 0$. Here $\alpha \in (0, 1)$ is a fixed value, and π is a log-normal distribution $\mathcal{LN}(0, \sigma^2)$. SMPE can be showed to exists in the linear form: $h_k(s) = ks$ with $k > 0$. Then the capital stock $(s_t)_{t=1}^{\infty}$ obeys the formula:

$$s_{t+1} = \omega_t (1-k)^{\alpha} s_t^{\alpha} \quad \textit{with} \quad s_1 = s,$$

or after substitution $y_t := \ln(s_t)$, $\epsilon_t := \ln(\omega_t)$

$$y_{t+1} = \alpha \ln(1-k) + \alpha y_t + \epsilon_t. \quad \textit{with} \quad y_1 = \ln(s). \tag{7}$$

Clearly ϵ_t is an i.i.d. process with $\mathcal{N}(0, \sigma^2)$ distribution. The continuation value function is

$$v^k(s) = \mathbb{E}\left(\sum_{t=1}^{\infty} \ln(s_t)\delta^{t-1}\right) = \mathbb{E}\left(\sum_{t=1}^{\infty} y_t \delta^{t-1}\right).$$

One computes v^k the following way:

$$v^k(s) = A_k \ln(s) + B_k \tag{8}$$

for some $A_k, B_k > 0$. These constants can be found by the following program:

- *find v^k as a solution of Bellman equation:*

$$v^k(s) = \ln(ks) + \delta \int_S v^k(s') Q(ds'|(1-k)s); \tag{9}$$

- *verify whether v^k obeys the **transversality condition**:*

$$\limsup_{t\to\infty} v^k(s_t)\delta^{t-1} \leq 0$$

  for almost all realizations of s_t starting from s, and if

$$\limsup_{t\to\infty} v^k(s_t)\delta^{t-1} < 0$$

  then $v_t(s) = -\infty$ (see Wiszniewska-Matyszkiel (2011); Wiszniewska-Matyszkiel and Singh (2020) or Kamihigashi (2008) for details).

Substituting v^k from (8) into (9) one obtains

$$A_k \ln(s) + B_k = (1 + \alpha\delta A_k)\ln(s) + \ln(k) + \delta\alpha A_k \ln(1-k) + \delta B_k.$$

Hence

$$A_k = \tfrac{1}{1-\alpha\delta}, \quad \textit{and} \quad B_k = \frac{ln(k) + \frac{\delta\alpha\ln(1-k)}{1-\alpha\delta}}{1-\delta}.$$

As a result,

$$v^k(s) = \frac{1}{1-\alpha\delta}\ln(s) + \frac{ln(k) + \frac{\delta\alpha\ln(1-k)}{1-\alpha\delta}}{1-\delta}.$$

Now one needs to verify the transversality conditions. For all t one obtains:

$$v^k(s_t) = \frac{1}{1-\alpha\delta}\ln(s_t) + \frac{ln(k) + \frac{\delta\alpha\ln(1-k)}{1-\alpha\delta}}{1-\delta} = \frac{1}{1-\alpha\delta}y_t + B_k, \tag{10}$$

hence

$$\delta^{t-1}v^k(s_t) = \frac{\delta^{t-1}y_t}{1-\alpha\delta} + B_k\delta^{t-1}. \tag{11}$$

The expression in (11) tends to 0 *provided the series* $\sum_{t=1}^{\infty} y_t\delta^{t-1}$ *are convergent. Let* $z_t := y_t\delta^{t-1}$*. By (7) the following is true:*

$$z_{t+1} = \alpha\ln(1-k)\delta^t + \alpha\delta z_t + \delta^t\epsilon_t.$$

Hence the expectation $\mathbb{E}(z_t)$ *satisfies the following difference equations*

$$\mathbb{E}(z_{t+1}) = \alpha\ln(1-k)\delta^t + \alpha\delta\mathbb{E}(z_t).$$

Hence $\mathbb{E}(z_t)$ *satisfies*

$$\mathbb{E}(z_t) = \frac{\alpha\ln(1-k)}{\delta(1-\alpha)}\delta^t + \alpha^t\delta^t\ln(s),$$

Moreover, $Var(z_t)$ *satisfies*

$$Var(z_{t+1}) = \alpha^2\delta^2 Var(z_t) + \delta^{2t}\sigma^2$$

hence

$$Var(z_t) = \sigma^2(\alpha\delta)^{2t} + \frac{\sigma^2}{\delta^2(1-\alpha^2)}.$$

As a result, both series $\sum_{t=1}^{\infty}\mathbb{E}(z_t)$ *and* $\sum_{t=1}^{\infty}Var(z_t)$ *are convergent. Kolmogorov two-sequence Theorem suffices to conclude the series* $\sum_{t=1}^{\infty} z_t$ *converges almost surely, consequently* z_t *tends to* 0 *almost surely. Hence, by definition of* z_t*, the transversality condition holds, and so* v^k *is (10) defines the continuation value function. Then, the current self maximizes*

$$u(c) + \beta\delta\int_S v^k(s')Q(ds'|s-c) = \ln(s) + \frac{\beta\delta}{1-\alpha\delta}\ln(s-c) + \delta B_k$$

such that $c \in [0, s]$*. The solution is SMPE and it is*

$$h^*(s) = \frac{1}{1+\frac{\beta\delta}{1-\alpha\delta}}s.$$

Apart from specific examples, however, the existence of equilibrium under general assumptions (for u unbounded below or Q possessing atoms) is an open question. Conditions sufficient to prove SMPE existence are now provided.

Assumption 1 *Assume that*

- *The utility function u is strictly concave, increasing, and $u(0) = 0$;*
- *The transition function obeys the following properties:*
  - *For $i \in S$, the transition $Q(S|i, s)$ does not depend on particular $s \in S$, call it $Q(\cdot|i)$;*
  - *For $i \in S \setminus \{0\}$, the transition $Q(\cdot|i)$ is a nonatomic measure, and $Q(\cdot|\{0\})$ is either nonatomic or $Q(\{0\}|0) = 1$;*
  - *$Q(\cdot|i)$ is stochastically increasing, that is for any increasing, Borel and bounded function $f : S \mapsto \mathbb{R}$ the function*

$$i \in S \mapsto \int_S f(s')Q(ds'|i).$$

    is increasing.

These assumptions are rather standard. Nonatomicity of $Q(\cdot|i)$ is assumed to guarantee existence of SMPE is a general class of models. Two remarks are in order.

Remark 1 *A typical example of u on S is the power function $u(c) = c^\alpha$, for $\alpha \in (0, 1)$. Other functions satisfying the assumptions are e.g. $u(c) = \ln(1+c)$, $u(c) = 1 - e^{-c}$. For particular choice of S, for example $S = [0, 1]$, one may take $u(c) = 2c - c^2$.*

Remark 2 *A typical example of the transition probability in economic growth theory is $s_{t+1} = F(\omega_{t+1}, i_t)$, where ω_{t+1} is a independent, identically distributed shock, F is a continuous function in i and a Borel function in ω.*

Apart from verifying SMPE existence one would like to provide some characterization of SMPE policies. Therefore, restrict attention to a class of investment policies that are increasing and lower semicontinuous:

$$\mathcal{I} := \{g \in \mathcal{H} : g \text{ is increasing and lower semicontinuous}\}. \tag{12}$$

Then the corresponding class of consumption policies is:

$$\mathcal{G} := \{h \in \mathcal{H} : h(s) = s - g(s) \text{ for all } s \in S, \text{ and some } g \in \mathcal{I}\}.$$

The next theorem assures SMPE existence and provides characterization of investment policies (namely, that they are increasing and lower semicontinuous).

Theorem 1 (Existence of SMPE) *Under Assumption 1 there exist a SMPE in $\mathcal{G}$.*

Proof: Sketch of the proof. Endow $\mathcal{G}$ with the *weak topology* with the convergence $\to^*$. Recall, this topology restricted to $\mathcal{G}$ is metrizable and the convergence $\to^*$ is equivalent to the following condition: $h_n \to^* h$ as $n \to \infty$ if and only if $h(s_n) \to h(s)$ as $n \to \infty$ whenever $s_n \to s$ as $n \to \infty$, and s is a continuity point of h. The set $\mathcal{G}$ is homeomorphic with the set of probability measures by the following transformation:

$$h \in \mathcal{G} \mapsto \eta_h,$$

where η_h is a probability measure whose cumulative distribution function is $s - h(s)$ i.e., $\eta_h([0, s)) = s - h(s)$ for $s \leq \bar{s}$ and $\eta_h([0, s)) = 1$ for $s > \bar{s}$. Hence $\mathcal{G}$ is compact. The weak topology is embedded into a topological vector space $\mathbf{G}$ of signed measures with locally bounded variation (see Jaśkiewicz and Nowak (2022) for details). For any $h \in \mathcal{G}$ define

$$br(h)(s) := \max \arg \max_{c \in [0,s]} P(c, s; h).$$

Clearly $br(h) \in BR(h)$. One only needs to show there exists a fixed point of $h \mapsto br(h)$. The operator br is well defined and maps (compact) $\mathcal{G}$ into itself (Lemma 3.2 in Balbus et al. (2015a)). Moreover, br is continuous (see proof of Theorem 1 in Balbus et al. (2015a)). Hence by the Schauder-Tychonoff fixed point theorem, br has a fixed point in $\mathcal{G}$. This fixed point is SMPE. ■

UNIQUENESS OF SMPE. SEARCHING THROUGH CONTINUATION VALUES

The generalized Bellman operator T along with some additional structure on the model can be used to show uniqueness of SMPE. Clearly, to obtain such a strong result for $\beta < 1$ one needs new assumptions, in particular on the transition Q. In this section, allow the stochastic transition $Q(\cdot|i,s)$ to depend on s, and relax the non-atomicity of Q, but at the cost of imposing certain "mixing" condition that will be clear in the assumption:

Assumption 2 *Assume*

- *u is strictly concave and increasing and $u(0)=0$;*
- *transition probability Q has the following form*

$$Q(\cdot|i,s) = p(\cdot|i,s) + (1 - p(S|i,s))\delta_0(\cdot),$$

  where $\delta_0(\cdot)$ is a unit point mass (Dirac delta) concentrated in 0 and

  - *$p(\cdot|i,s)$ is a finite measure such that for any $s>0$ and $i \in [0,s]$, $p(S|i,s) < 1$, and $p(\{0\}|0,0) = 1$;*
  - *for every bounded and Borel measurable function f such that $f(0)=0$,*

$$i \in S \mapsto \int_S f(s')Q(ds'|i,s) \tag{13}$$

    is increasing and concave in i and continuous in (i,s).

Examples of primitive data that satisfy Assumption 2 will be provided later in example 3. Now, instead, some consequences of Assumption 2 on the operator T are provided. For any $s \in S$, $v \in \mathcal{V}$ define

$$\Pi(c,s,v) = u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s).$$

Observe that under Assumption 2 $\Pi(\cdot, s; h)$ is strictly concave regardless on $s \in S$ and $v \in \mathcal{V}$. Since

$$B(v)(s) = \arg\max_{c\in[0,s]} \Pi(c,s;v),$$

hence $B(v)(s)$ is a singleton. Consequently the generalized Bellman operator T is well defined. We claim that T is an increasing operator.

Claim 1 *T is an increasing operator on $\mathcal{V}$.*

Proof: Clearly A is increasing. We show that B is decreasing. For proving the last assertion, apply the standard Topkis Theorem (see Theorem 6.1 in Topkis (1978)). For this purpose show, for any s, $(c, v) \in [0, s] \times \mathcal{V} \mapsto \Pi(c, s; v)$ has decreasing difference. For $s \in S$, let $c_1 < c_2 \leq s$ and define

$$\kappa(\cdot) := p(\cdot|s - c_1, s) - p(\cdot|s - c_2, s).$$

Clearly κ is a measure. Moreover, for any $v \in \mathcal{V}$

$$\int_S v(s')Q(ds'|s - c_2, s) - \int_S v(s')Q(ds'|s - c_1, s) = -\int_S v(s')\kappa(ds').$$

Hence

$$\Pi(c_2, s, v) - \Pi(c_1, s, v) = u(c_2) - u(c_1) - \beta\delta \int_S v(s')\kappa(ds')$$

is decreasing in v. As a result, for any s, $(c, v) \in [0, s] \times \mathcal{V} \mapsto \Pi(c, s; v)$ has decreasing differences. Hence by the standard Topkis (1978) Theorem, $B(v)$ is decreasing in v. Hence T increases with v for any $\beta \in [0, 1]$ as A is increasing with v. ■

Monotonicity of T happens to be critical to assure SMPE uniqueness. Again, apart from just proving SMPE uniqueness, we provide some characterization of equilibrium policies. To do so, define a class of policies where both consumption and investment policies are increasing:

$$\mathcal{L} := \{h \in \mathcal{H} : \text{both } h(s) \text{ and } s - h(s) \text{ are increasing (in } s)\}.$$

By Theorems 1 and 2 in Balbus et al. (2018) one obtains the following result:

Theorem 2 (Uniqueness and attracting of SMPE) *Under Assumption 2 there exists a unique continuation value v^*, and corresponding unique SMPE. Moreover, for any initial $v_0 \in \mathcal{V}$ the sequence of successive iterations $v_{t+1} = T(v_t)$ for $t \geq 0$ tends uniformly to v^*. That is*

$$\lim_{t \to \infty} ||v_t - v^*||_\infty = 0. \tag{14}$$

Let h^* be the SMPE. If $Q(\cdot|i, s)$ does not depend on s, then $h^* \in \mathcal{L}$.

Proof: Sketch of the proof. Obviously $\mathcal{V}$ is a Banach space, and it is routine to verify that $T : \mathcal{V} \mapsto \mathcal{V}$. Observe that v^* corresponding to a MPE is a fixed point of the well defined operator T. By Claim 1, T is an increasing operator. On the other hand, for any nonnegative constant k,

$$T(v+k)(s) = T(v)(s) + \delta k.$$

Indeed, $B(v+k)(s) = B(v)(s)$ and $A(v+k)(s) = A(v)(s) + \beta\delta k$. By Blackwell Theorem (see Stokey et al. (1989)) we conclude T is a δ-contraction mapping. As a result, the fixed point v^* of T is uniquely determined and satisfies (14). Moreover, this is a unique continuation value function for a SMPE. Since $h^* = B(v^*)$, and B is well defined, hence SMPE is uniquely determined.

Finally, note that when $Q(\cdot|i,s)$ does not depend on s the function $(c,s) \mapsto \Pi(c,s,v^*)$ has increasing differences. Hence by Topkis (1978) h^* is increasing then $(i,s) \mapsto \Pi(s-i,s,v^*)$ have increasing differences. Then, $h^* \in \mathcal{L}$. ■

Example 3 provides a class of transition functions that obeys Assumption 2.

Example 3 *Typical example is a linear combination between some probability measure and a unit point mass of absorbing state. For example*

$$Q(\cdot|i,s) = \sum_{l=1}^{L} g_l(i)\lambda_l(\cdot|s) + \left(1 - \sum_{l=1}^{L} g_l(i)\right)\delta_0(\cdot).$$

Here any of $g_l : S \mapsto [0,1]$ *is increasing and concave such that* $g_l(0) = 0$ *and* $\sum_{l=1}^{L} g_l(i) \leq 1$ *for all* $i \in S$. *This class of probability distribution was used in Rogerson (1985), Amir (1996), Szajowski (2006), Balbus and Nowak (2008) and Balbus et al. (2013b), Balbus et al. (2015b) among others. If one relaxes this assumption, only existence of equilibrium is assured.*

MONOTONE COMPARATIVE STATICS

With Assumption 2 in place, one can prove the monotone comparative statics result for equilibrium policies. Begin by parameterizing the transition Q by some parameter $\theta \in \Theta$. Parameter θ can specify for example economy's productivity (e.g. the higher the θ the less productive the economy is), but could have other economic interpretations. Now, for $s \in S$, $c \in [0,s]$, $v \in \mathcal{V}$ and $\theta \in \Theta$ define

$$\Pi(c,s;v,\theta) := u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s,\theta).$$

Furthermore, define the operators A_θ, B_θ and T_θ as the natural (parameterized) generalizations of A, B and T:

$$A_\theta(v)(s) = \max_{c\in[0,s]} \Pi(c, s; v, \theta),$$

$$B_\theta(v)(s) = \arg\max_{c\in[0,s]} \Pi(c, s; v, \theta),$$

and

$$T_\theta(v)(s) = \frac{1}{1-\beta} A_\theta(v)(s) - \frac{1-\beta}{\beta} u(B_\theta(v)).$$

The following assumption specifies, how the change of parameter θ affects primitives of the model.

Assumption 3 *Let us assume:*

- *u does not depend on θ and obeys assumption 2;*
- *for any $s, i \in S$ and $\theta \in \Theta$ let $Q(\cdot|i, s, \theta) = p(\cdot|i, s, \theta) + (1 - p(S|i, s, \theta))\delta_0(\cdot)$, where for each θ $p(\cdot|i, s, \theta)$ obeys Assumption 2;*
- *for each $v \in \mathcal{V}$ function $(i, \theta) \to \int_S v(s')p(ds'|i, s, \theta)$ has decreasing differences with (i, θ) and $\theta \to \int_S v(s')p(ds'|i, s, \theta)$ is decreasing on Θ.*

By Assumption 3 for any θ, the model obeys Assumption 2. As a result, there exists a unique SMPE h^*_θ and its continuation value v^*_θ. Moreover, for any $s \in S$,

$$v^*_\theta(s) = T_\theta(v^*)(s) \quad \text{and} \quad h^*_\theta(s) \in B_\theta(v^*)(s).$$

In fact, both functions increase in θ as a result of the following claim.

Claim 2 *Assume 2. Then, the function $B_\theta(v)$ decreases in θ and $T_\theta(v)$ decreases in $\theta \in \Theta$ and increases in $v \in \mathcal{V}$.*

Proof: The the third bullet of Assumption 3 it follows that for any $v \in \mathcal{V}$

$$\theta \in \Theta \mapsto \int_S v(s')p(ds'|i, s, \theta)$$

is decreasing, hence $A_\theta(v)$ is decreasing in θ. Obviously it is also increasing in v. For any $s \in S$ and for have for $c_1 < c_2 \leq s$

$$\Pi(c_2, s; v, \theta) - \Pi(c_1, s; v, \theta) := -\int_S v(s')\kappa_\theta(ds')$$

where

$$\kappa_\theta(\cdot) := p(\cdot|s - c_1, s, \theta) - p(\cdot|s - c_2, s, \theta).$$

Obviously, κ_θ is a measure hence is increasing in v. Moreover, by the third bullet in Assumption 3, is increasing in θ. Hence Π has decreasing differences in (s, v) and increasing differences in (s, θ). By the standard Topkis (1978) Theorem, $B_\theta(v)(s)$ is decreasing in v and increasing in θ. As a result, $T_\theta(v)$ is decreasing in θ and increasing in v. ■

This suffices to state the SMPE monotone comparative statics theorem:

Theorem 3 (Monotone comparative statics) *Let Assumption 3 be satisfied. Then, the equilibrium policy $\theta \to h^*_\theta$ is increasing, and the continuation value $\theta \to v^*_\theta$ is decreasing.*

Proof: For any $\theta \in \Theta$, let $v_\theta : S \mapsto \mathbb{R}$ be a Borel function. Suppose that $\theta \mapsto v_\theta(s)$ is decreasing for any $s \in S$. By Claim 2, it follows that $T_\theta(v_\theta)$ is decreasing. Hence using the standard induction, one can show that the n-fold decomposition $T^n_\theta(\mathbf{0})$ decreases in θ. Indeed, for $n = 1$ it is clear. Suppose it is true for n and $T^n_\theta(\mathbf{0})$ decreases in θ. Then, by Claim 2 and induction hypothesis,

$$T^{n+1}_\theta(\mathbf{0}) = T_\theta(T^n_\theta(\mathbf{0}))$$

it is decreasing in θ. Consequently, $T^n_\theta(\mathbf{0})$ is decreasing in θ for any n. By Theorem 2, $v^*_\theta = \lim_{n\to\infty} T^n(\mathbf{0})$ decreases in θ as well. Hence, applying Claim 2 again one concludes $h^*_\theta = B_\theta(v^\theta)$ is increasing. ■

The theorem allows to conduct monotone comparative statics exercise with respect to parameter affecting transition Q. Similar approach can be used to obtain more specific results on comparative statics with respect to parameter β. Higher β will correspond to a case of lower θ, i.e. more productive Q. It is illustrated by the following example.

Consider a pair of modified operators:

$$\hat{A}(v)(s) = \max_{c\in[0,s]} \left\{ u(c) + \delta \int_S v(s')Q(ds'|s - c, s) \right\},$$

$$\hat{B}(v)(s) = \arg\max_{c\in[0,s]} \left\{ u(c) + \delta \int_S v(s')Q(ds'|s - c, s) \right\}.$$

Observe that:

$$\hat{A}(\beta v) \equiv A(v)$$
$$\hat{B}(\beta v) \equiv B(v).$$

We claim that $\hat{v}^*$ is a fixed point of operator $\hat{T}$ if and only if $v^* = \frac{\hat{v}^*}{\beta}$ is a fixed point of T, where:

$$\hat{T}(v) = \hat{A}(v)(s) - (1-\beta)u(\hat{B}(v)(s))$$

Now consider a parameterized fixed point problem: $\hat{T}_\beta(v) = \hat{A}v(s) - (1-\beta)u(\hat{B}(v)(s))$. Observe that for each v operator $\hat{T}_\beta(v)$ is increasing in β (recall $\hat{A}$ and $\hat{B}$ do not depend on β). Under assumption 2 $\hat{T}_\beta$ is a monotone contraction. Hence the unique fixed point is increasing in β. This is summarized in the next claim.

Claim 3 *(Monotone comparative statics in impatience parameter) Under assumption 2 equilibrium consumption $h^*_\beta = \hat{B}(\hat{v}^*_\beta) = B(v^*_\beta)$ is decreasing and, analogously, equilibrium investment is increasing in β.*

GENERALIZED EULER EQUATIONS

Beginning with the work of Harris and Laibson (2001), many researchers have applied the so-called "generalized Euler equation" approach to solving dynamic/stochastic games. It is possible in our setting to provide conditions on the primitives of the model such that the existence of a unique *differentiable* pure strategy Markovian equilibrium can be guaranteed. This allows one to state the version of the generalized Euler equation that actually characterizes such a Markovian equilibrium. Suppose that $Q(\cdot|i,s)$ does not depend on s and denote it by $Q(\cdot|i)$. Similarly denote $p(\cdot|i)$

Assumption 4 *Assume 2. Moreover assume the primitives of the model are sufficiently smooth.*[6] *In particular let $p(\cdot|i)$ has a density:*

$$p(S_0|i) = \int_{S_0} q(s'|i)ds',$$

for any Borel set $S_0 \subset S$, and its derivative is denoted by:

$$q'(s'|i) := \frac{d}{di}q(s'|i);$$

[6] See Balbus et al. (2022) for details.

Put

$$D(s) = \frac{d}{di} \int_S v^*(s')Q(ds'|i)\Big|_{i=s-h^*(s)}. \tag{15}$$

Theorem 4 (Generalized Euler Equations) *Assume 4. Then, any interior h^* is differentiable and the following equation is satisfied for any $s > 0$:*

$$\begin{aligned} \frac{u'(h^*(s))}{\beta\delta} = D(s) &= \left(\frac{1}{\beta} - 1\right) \int_S u'(h^*(s)))(h^*)'(s')q'(s'|s - h^*(s))ds' \\ &\quad -\frac{1}{\beta} \int_S u'(h^*(s'))q'(s'|s - h^*(s))ds'. \end{aligned} \tag{16}$$

Proof: Now derive the *Generalized Euler Equation*. Observe that interior h^* satisfies the First Order Condition for any s:

$$u'(h^*(s)) - \beta\delta D(s) = 0. \tag{17}$$

Using the Fundamental Theorem of the Integral Calculus for Riemann-Stieltjes integrals (see Hewitt and Stromberg (1965) Theorem 18.19 or Amir (1997), Theorem 3.2) one obtains:

$$\frac{d}{di} \int_S v^*(s')Q(ds'|i)\Big|_{i=s-h^*(s)} = -\int_S (v^*)'(s')q'(s'|s - h^*(s))ds'. \tag{18}$$

Then differentiating v^*:

$$(v^*)'(s) = u'(h^*(s)))(h^*)'(s) + \delta D(s)(1 - (h^*)'(s)).$$

Hence

$$\begin{aligned} \int_S (v^*)'(s)q'(s|i)ds &= \int_S u'(h^*(s))(h^*)'(s)q'(s|i)ds \\ &\quad +\delta \int_S D(s)(1 - (h^*)'(s))q'(s|i)ds. \end{aligned}$$

Substituting $i = s - h^*(s)$ above, and applying (18) one obtains:

$$\begin{aligned} -D(s) &= \int_S u'(h^*(s'))(h^*)'(s')q'(s'|s - h^*(s))ds' \\ &\quad +\delta \int_S D(s')(1 - (h^*)'(s'))q'(s'|s - h^*(s))ds'. \end{aligned}$$

Hence and by (17)

$$\begin{aligned} -D(s) &= \int_S u'(h^*(s')))(h^*)'(s')q'(s'|s - h^*(s))ds' \\ &\quad +\frac{1}{\beta} \int_S u'(h^*(s'))(1 - (h^*)'(s'))q'(s'|s - h^*(s))ds', \end{aligned}$$

or equivalently (16). ■

Clearly, replacing $(h^*)'(s) = 1 - (g^*)'(s)$, were g^* denote equilibrium investment one obtains the generalized Euler equation:

$$\begin{aligned}\frac{u'(h^*(s))}{\beta\delta} &= \left(1 - \frac{1}{\beta}\right) \int_S u'(h^*(s)))(g^*)'(s')q'(s'|g^*(s))ds' \\ &\quad - \int_S u'(h^*(s'))q'(s'|g^*(s))ds'.\end{aligned}$$

EXTENSIONS

The focus of this chapter has been on the case in which S is a bounded interval. Using the results from Balbus et al. (2020), one can conclude that the thesis of Theorem 1 are satisfied for "weightly-bounded" utility functions. Then, applying the main result in Balbus et al. (2018), one deduces that the thesis of Theorem 2 is also valid in case of "bounded by sequence" utility functions. The definition of "weightly-bounded" function is formalized as follows.

Definition 5 (w-bounded function) *Let $w : S \mapsto (0, \infty)$. A function v is w-bounded if there exists $M > 0$ such that for any $s \in S$, $|v(s)| \leq M\, w(s)$.*

The space of w-bounded functions is a Banach space with the natural norm

$$||v||_w := \sup_{s \in S} \left| \frac{v(s)}{w(s)} \right|.$$

Another approach is a "boundedness by a sequence". Let $(S_j)_{j=1}^{\infty}$ be a sequence of Borel subsets of S such that any of S_j has non-empty interior and $S_1 \subset S_2 \subset \dots$, and let $\mathbf{m} := (m_j)_{j=1}^{\infty}$ be a strictly monotone sequence of positive numbers such that

$$r := \sup_{j \in \mathbb{N}} \frac{m_{j+1}}{m_j} < \infty.$$

The formal definition of "bounded by a sequence" is now provided.

Definition 6 (m-bounded) *The function v is $\mathbf{m}$-bounded if for any $j \in \mathbb{N}$, $||v||_j := \sup_{s \in S_j} |v(s)| \leq m_j$.*

Let us focus attention on the set of Borel measurable functions. The seminorms $||v||_j$ define a locally convex space of functions bounded on all S_j. The set of $\mathbf{m}$-bounded functions is embedded into a larger space

$$\mathbf{V} := \left\{ v : S \mapsto \mathbb{R}, v \text{ is Borel}, ||v||_j < \infty, \text{ for all } j, \text{ and } \sum_{j=1}^{\infty} \frac{||v||_j}{m_j} \zeta^{j-1} < \infty \right\}$$

for some $\zeta \in (0, 1)$. Clearly $\mathbf{V}$ is a vector space with the following norm

$$||v|| := \sum_{j=1}^{\infty} \frac{||v||_j}{m_j} \zeta^{j-1}.$$

The set of generic elements of such functions is:

$$\mathcal{V}^{\mathbf{m}} := \{v \in \mathbf{V} : v(0) = 0, ||v||_j \leq m_j \text{ for all } j \in \mathbb{N}\}$$

Clearly $\mathcal{V}^{\mathbf{m}} \subset \mathbf{V}$. Due to Matkowski and Nowak (2011) the following is true:

Proposition 1 *The tuple* $(\mathbf{V}, || \cdot ||)$ *defines a Banach space and* $\mathcal{V}^{\mathbf{m}}$ *is a closed subset of* $\mathbf{V}$.

Rincon-Zapatero and Rodriguez-Palmero (2003) provide the following definition:

Definition 7 (1-local contraction) *The operator* $\Phi : \mathcal{V} \mapsto \mathcal{V}$ *is a 1-local contraction if there is a constant* $\xi \in (0, 1)$ *such that*

$$||\Phi(v_1) - \Phi(v_2)||_j \leq \xi ||v_1 - v_2||_{j+1}.$$

Rincon-Zapatero and Rodriguez-Palmero (2003, 2009)[7] show the following:

Proposition 2 *Let* $\Phi : \mathcal{V}^{\mathbf{m}} \mapsto \mathcal{V}^{\mathbf{m}}$ *be a 1-local contraction with a constant* $\xi \in (0, 1)$ *and suppose* $\xi r < \zeta$. *Then* Φ *is a contraction mapping with respect to the metric induced by the norm* $|| \cdot ||$ *and its modulus is* $\frac{\xi r}{\zeta}$.

It is now useful to consider assumptions for both approaches: "weightly-bounded" as well as "bounded by sequence" when studying the existence of SMPE.

[7] See also Matkowski and Nowak (2011).

WEIGHTED BOUNDED FUNCTIONS

Start with the assumption in case of weighted bounded felicity function.

Assumption 5 (w-bounded approach) *Suppose that Assumption 1 holds. Moreover, there exists a Borel function $w : S \mapsto [1, \infty)$ such that:*

- *The felicity function satisfies $||u||_w \leq \bar{u}$;*
- *The transition function satisfies*
$$\bar{Q} := \sup_{s \in S} \sup_{i \in [0,s]} \frac{\int_S w(s')Q(ds'|i)}{w(s)} < \infty$$
and
$$\delta \bar{Q} < 1.$$

Let Δ_n be the set of all non-negative Borel measures on $[0, n]$ such that any of $\eta \in \Delta_n$ satisfies $\eta([0, n]) \leq n$. Let $\Delta := \prod_{n=1}^{\infty} \Delta_n$. Endow Δ_n with the standard weak topology, and Δ with the product topology. Let $\mathcal{H}$ and $\mathcal{G}$ be as in (3) and respectively (12). Endow $\mathcal{G}$ with adapted weak topology i.e. $h_n(s) \to^w h(s)$ as $n \to \infty$ whenever h is continuous at s. There is an isomorphism between $\mathcal{I}$ and a subset of product of measures on Δ. We provide the construction by Balbus et al. (2020). Any $h \in \mathcal{I}$ induces a unique element $(\eta_n)_{n=1}^{\infty} \in \Delta$ such that

$$h \in \mathcal{G} \mapsto \eta_n([0, s]) = s - h(s), \text{for } s \in [0, n].$$

In other words, $s - h(s)$ is a cumulative distribution function of $\eta_n(\cdot)$ on $[0, n]$.

Let
$$M_0 := \frac{\bar{u}}{1 - \delta \bar{Q}}$$
and let s_t be a Markov chain generated by a policy $h \in \mathcal{I}$, i.e. with the transition probability $Q(\cdot|h(s))$. Then for any t,

$$\mathbb{E}_s(u(h(s_t))) \leq \mathbb{E}_s(u(s_t)) \leq \bar{u}\mathbb{E}_s(w(s_t)).$$

Then
$$\mathbb{E}_s \delta^t w(s_{t+1}) = \mathbb{E}_s \left(\delta^t \int_S w(s')Q(ds'|s_t - h(s_t)) \right) \leq \delta \bar{Q} \mathbb{E}_s \delta^{t-1} w(s_t).$$

Hence V_h is well defined w-bounded function, and

$$V_h(s) \leq \mathbb{E}_s \left(\sum_{t=1}^{\infty} w(s_t) \delta^{t-1} \right) \leq \frac{\bar{u}}{1 - \delta \bar{Q}} w(s) = M_0 w(s).$$

Then, one constructs SMPE h^* as a fixed point of $br(h)$, which is continuous and maps $\mathcal{I}$ into itself (Lemma 13 in Balbus et al. (2020)). Hence the thesis of Theorem 1 are satisfied.

FUNCTIONS BOUNDED BY A SEQUENCE

Using the results in Balbus et al. (2018), the thesis of Theorem 2 in case of $\mathbf{m}$-bounded utility functions on S are valid. The following assumptions is needed:

Assumption 6 (m-bounded approach) *Assume 2. Moreover, suppose there exists a monotone sequence* $\mathbf{m} = (m_j)_{j=1}^{\infty}$ *such that* $m_1 > 0$ *and* $r = \sup_{j \in \mathbb{N}} \frac{m_{j+1}}{m_j} < \infty$, *and an ascending sequence of sets* $(S_j)_{j=1}^{\infty}$ *such that* $S_j \subset S$ *and such that*

- *for any* j, $\sup_{s \in S_j} u(s) = (1-\beta)m_j$;
- *for any* j, $p(S_{j+1}|i,s) = p(S|i,s) = 1$ *whenever,* $s \in S_j$ *and* $i \in [0,s]$;
- *the sequence* $\mathbf{m}$ *satisfies*
$$\delta \sup_{j \in \mathbb{N}} \frac{m_{j+1}}{m_j} \leq \beta.$$

In other words, $Q(S_{j+1}|i,s) = 1$ whenever $s \in S_j$, that is the state visits sequentially $S_1, S_2, \ldots$ almost surely. Recall the definitions of $A(v)$, $B(v)$ and $T(v)$. The operator T maps $\mathcal{V}^{\mathbf{m}}$ into itself. Indeed, for $s \in S_j$

$$\begin{aligned} T(v)(s) &= u(B(v)(s)) + \delta \int_S v(s')Q(ds'|s - B(v)(s)) \\ &\leq (1-\beta)m_j + \delta m_{j+1} \leq \left(1 - \beta + \beta \sup_{j \in \mathbb{N}} \frac{m_{j+1}}{m_j}\right) m_j \leq m_j. \end{aligned}$$

Furthermore, T is a 1−local contraction with δ on $\mathcal{V}^{\mathbf{m}}$, hence and by Proposition 2 it is a contraction mapping with the metric induced by the norm on $||\cdot||$ on $\mathcal{V}^{\mathbf{m}}$. $(\mathcal{V}^{\mathbf{m}}, ||\cdot||)$ is a complete metric space, hence T has a unique fixed point in T. Hence the thesis of Theorem 2 are satisfied.

RECURSIVE UTILITY AND GENERAL CERTAINTY EQUIVALENTS

Standard dynamic models rely on the assumption that the utility functions of successors defined over sequences of random levels of consumption are represented by a time additive expected overall utility, which discounts future temporal utilities at a constant rate.

This approach cannot explain some problems in economics. For example, the standard discounted utility cannot explain the *equity premium puzzle* postulated by Mehra and Prescott (1985), and the standard expectation cannot explain the Allais and Ellsberg paradoxes, see e.g. Chew (1983), Dekel (1986), Chew and Epstein (1989). To remedy these issues, some authors propose other approaches based on recursive utility and general certainty equivalents. It is best to start our discussion with some basic introduction to these two generalizations and then provide its application to quasi-hyperbolic discounting.

RECURSIVE UTILITY AND CERTAINTY EQUIVALENTS: BASIC DEFINITIONS AND EXAMPLES

Let us start with the standard deterministic discounted utility function for the self t

$$V_t = \sum_{\tau=t}^{\infty} u(c_\tau)\delta^{\tau-t}$$

that obeys the Bellman equation. Koopmans (1960) provides the generalization of this approach toward the *recursive utility function*, i.e., V_t that satisfies the equation,

$$V_t = \mathcal{A}(c_t, V_{t+1}),$$

where $\mathcal{A}$ is a function called *aggregator*. In case of the standard utility, the aggregator has an affine form $\mathcal{A}(c_t, V_{t+1}) = u(c_t) + \delta V_{t+1}$

$$V_t = u(c_t) + \delta V_{t+1}.$$

If $(c_\tau)_{\tau=t}^{\infty}$ is a sequence of lotteries adapted to some filtration $(\mathscr{F}_\tau)_{\tau=t}^{\infty}$, then V_t is a $\mathscr{F}_t$-measurable random function. Kreps and Porteus (1978) provide the following model:

$$V_t = \mathcal{A}(c_t, \mathbb{E}_t V_{t+1}),$$

where $\mathbb{E}_t$ is the expectation conditioned by $\mathscr{F}_t$. Epstein and Zin (1989) generalized the utility substituting $\mathbb{E}_t$ by a more general *certainty equivalent operator*.

Let $\mathscr{L}^1$ be the equivalence class of real valued integrable functions endowed with an order.

Definition 8 (Certainty Equivalent Operator (COP)) *For any t, the operator $\mathcal{M}_t$ mapping $\mathscr{L}^1$ into $\mathscr{F}_t$- measurable function is called* certainty equivalent operator *if*

- $\mathcal{M}_t(\alpha) = \alpha$ *whenever* α *is constant;*
- $\mathcal{M}_t(\cdot)$ *is a monotone operator on* $\mathscr{L}^1$.

The recursive utility that connects the general aggregator approach and the certainty equivalent operator was introduced by Epstein and Zin (1989). Their model can be written in the following form:

$$V_t = \mathcal{A}(c_t, \mathcal{M}_t(V_{t+1})).$$

The recursive utility $(V_t)_t$ solving the above system of equations is *dynamically consistent*. It means that the following condition holds: for any $t \in \mathbb{N}$ let $c^t = (c_\tau)_{\tau=t}^{\infty}$ be a consumption stream, and let $\tilde{c}^t := (\tilde{c}_\tau)_{\tau=t}^{\infty}$ be another one. Then $V_t(c_t, c^{t+1}) \leq V_t(c_t, \tilde{c}^{t+1})$ if and only if $V_{t+1}(c^{t+1}) \leq V_{t+1}(\tilde{c}^{t+1})$. In other words, the decision maker never regrets the decision made yesterday. The basic problem is the existence and uniqueness of recursive utility function. The issue has the positive solution under distinct assumptions.[8]

The most common certainty equivalent operators are:

- *Entropic risk measure* postulated by Weil (1993);

$$\mathcal{M}_t(V_{t+1}) = -\frac{1}{r} \ln \left(\mathbb{E}_t \left(e^{-rV_{t+1}} \right) \right)$$

  for $r \neq 0$;

- *Kreps and Porteus measure* postulated in their (1978) paper:

$$\mathcal{M}_t(V_{t+1}) = \left(\mathbb{E}_t V_{t+1}^r \right)^{\frac{1}{r}}$$

  for $r > 0$.

Both postulated measures are special cases of *quasi-linear mean*:

$$\mathcal{M}_t(V_{t+1}) = \psi^{-1} \left(\mathbb{E}_t \psi(V_{t+1}) \right),$$

for some monotone and invertible function ψ (see Chew (1983) and Dekel (1986)). Another interesting operator is the so-called *max-min* operator defined by Gilboa and Schmeidler (1989), which can applied in the large body of robust control literature:

$$V_t := \min_{\theta \in \Theta} \mathbb{E}_t^{\theta} V_{t+1},$$

[8]For the survey on this topic, we refer the reader to Balbus (2020), Bäuerle and Jaśkiewicz (2018), Becker and Rincón-Zapatero (2021), Bich et al. (2018), Bloise and Vailakis (2018), Borovicka and Stachurski (2020), Jaśkiewicz et al. (2014), Le Van and Vailakis (2005), Marinacci and Montrucchio (2010), Martins-da Rocha and Vailakis (2010), Weil (1993) and the references therein.

here $\mathbb{E}_t^\theta$ is the parameterized expectation, and the value $\theta \in \Theta$ is unknown for the current self (see e.g. Balbus et al. (2015b)). Some certainty equivalent operators are not provided explicitly but have application due to useful properties. For example, the certainty equivalent by Gul (1991) reflects elation and disappointment aversion.

GENERALIZED QUASI-HYPERBOLIC DISCOUNTING AND CERTAINTY EQUIVALENTS

The $\beta - \delta$ problem can be generalized in a number of directions, and we discuss one such generalization here. Let each self t have the utility

$$W_t = \mathcal{A}_1(c_t, \mathcal{M}_t(V_{t+1})),$$

where $(V_t)_{t=1}^\infty$ is the sequence of recursive utility satisfying

$$V_t = \mathcal{A}_2(c_t, \mathcal{M}_t(V_{t+1})).$$

Here $\mathcal{A}_1$ is an aggregator connecting present consumption and the expected utility of the successor. In turn, $\mathcal{A}_2$ is an aggregator connecting a consumption of future self and the expected utility the its successor. For example, $\beta - \delta$ problem has this form with

$$\mathcal{A}_1(c_t, \mathbb{E}_t V_{t+1}) = u(c_t) + \beta\delta\mathbb{E}_t V_{t+1}, \quad \text{and} \quad \mathcal{A}_2(c_t, \mathbb{E}_t V_{t+1}) = u(c_t) + \delta\mathbb{E}_t V_{t+1}$$

and the standard expectation. More generally, one can substitute the expectation operator by the following risk measure:[9]

$$\mathcal{M}_t = -\ln\left(\mathbb{E}_t(e^{-V_{t+1}})\right).$$

Policy h^* is SMPE if $h^*(s) \in \arg\max_{c\in[0,s]} P(c, s; h)$ for any $s \in S$ with

$$P(c, s; h) = u(c) - \beta\delta \ln \int_S e^{-V_h(s')} Q(ds'|s - c).$$

Here V_h solves the following Woodmans-Bellman equations

$$V_h(s) = u(h(s)) - \delta \ln \int_S e^{-V_h(s')} Q(ds'|s - h(s)).$$

Another possible generalization involves Koopmans et al. (1964) form. Let the $\mathcal{A}_1$ be as follows:

$$\mathcal{A}_1(c_t, V_{t+1}) = \ln(1 + c_t + \beta\delta V_{t+1}),$$

[9]In case of $\beta = 1$ the utility reduces to this studied in Bäuerle and Jaśkiewicz (2018) or in stochastic games by Asienkiewicz and Balbus (2019).

and let $\mathcal{A}_2$ be as follows

$$\mathcal{A}_2(c_t, V_{t+1}) = \sqrt{c_t^2 + \delta V_{t+1}},$$

and

$$\mathcal{M}_t(V_{t+1}) = \left(\mathbb{E}_t(V_{t+1}^3)\right)^{\frac{1}{3}}.$$

The current payoff has the form

$$P(c, s; h) = \ln(1 + c + \beta\delta\left(\int_S V_h(s')Q(ds'|s - c)\right).$$

Here V_h solves the Koopmans-Bellman equations:

$$V_h(s) = \sqrt{h^2(s) + \delta\left(\int_S V_h^3(s')Q(ds'|s - h(s))\right)^{\frac{1}{3}}}.$$

The generalized Bellman equation techniques can be easily adopted to cover these extensions. See Balbus et al. (2022).

MULTIDIMENSIONAL STATES

The generalized Bellman equation approach extends easily to the multi-dimensional state space. To see this, let the state space be $S = [0, \bar{s}] \subset \mathbb{R}^n$ and, for each $s \in S$, the action set $A(s) \subset A \subset \mathbb{R}^m$ for some A. Period utility is now $u : A \to \mathbb{R}_+$. Assumption 2 need to be generalized as follows:

Assumption 7 *Assume*

- *for each $s \in S$ set $A(s)$ is a compact and subcomplete sublattice with $A(0) = \{0\}$ moreover $s \mapsto A(s)$ is measurable,*
- *u is continuous, increasing and supermodular with $u(0) = 0$;*
- *transition probability $Q(\cdot|a, x) = p(\cdot|s, a) + (1 - p(S|a, x))\delta_0(\cdot)$, satisfies:*
  - *$p(\cdot|a, s)$ is a finite measure such that for any $s > 0$ and $a \in A(s)$, $p(S|i, s) < 1$, and $p(\{0\}|0, 0) = 1$;*
  - *for every bounded and Borel measurable function f such that $f(0) = 0$,*

$$a \mapsto \int_S f(s')Q(ds'|a, s) \tag{19}$$

is decreasing and supermodular in a and continuous in (a, s).

Under assumption 7 the generalized Bellman equation approach can still be used to analyze SMPE but now, as the $B(v)(s)$ can be multivalued, some of its selection need to be specified. Balbus et al. (2015c) use the greatest and the least selection, i.e. $\overline{B}(v)(s)$ and $\underline{B}(v)(s)$. This allows to specify two operators $\overline{T}$ and $\underline{T}$, respectively. As $B(v)$ is multivalued equilibrium uniqueness is not guaranteed but its existence and approximation can constructed by applying the following theorem.

Theorem 5 *Let assumption 7 hold. Then, the set of equilibrium continuation values possesses the least* $v^\star = \underline{T}(v^\star)$ *and the greatest* $w^\star = \overline{T}(w^\star)$ *elements corresponding to the greatest* $\overline{h}^\star$ *and the least* $\underline{h}^\star$ *SMPE.*

SELF-GENERATION APPROACH

In this last section inspired by the work of Abreu et al. (1990) or Mertens and Parthasarathy (1987), but adopted from the case of repeated games to dynamic games and short-memory equilibria (see also Doraszelski and Escobar (2012)), we discuss a self-generation approach to constructing a large set of non-stationary equilibria in this model. This method relies on the successive approximation of the sets of functions or equilibria value. The advantage of this method is relatively weak assumption on felicity functions and the transition probability. Moreover the method is useful when there are multiple equilibria.[10] The drawback is that this method allows to verify the existence of equilibria in a set of non-stationary strategies. In particular, after verifying the existence of the equilibria, we never know whether they are stationary or not, even if the model is time-invariant. In this subsection we briefly present the results by Balbus and Woźny (2016). Similar method can be found in non-stationary models as well. For example see Balbus et al. (2020), or in case of bequest games, see Balbus et al. (2017).

We come back to the standard infinite horizon model. For the main result in this section, the following assumptions are needed.

Assumption 8 *Assume 1 and moreover assume that for any* $s_0 \in S$ *the set* $Z_0(s_0) := \{i \in S : Q(\{s_0\}|i) > 0\}$ *is at most countable.*

This new assumption is not very restrictive, as is illustrated by the following example.

[10] Multiplicity of equilibria in quasi-hyperbolic discounting models was demonstrated by Krusell and Smith (2003), Bernheim et al. (2015) and Cao and Werning (2018) among others. See also an example with three SMPE in the closely related class of altruistic games in Balbus et al. (2013a).

Remark 3 *Observe that if $Q(\cdot|i)$ is nonatomic, then any of the set $Z_0(s_0)$ is empty and if $Q(\cdot|i)$ is deterministic such that $Q(A|i) = \mathbf{1}_A(G(i))$ for some strictly increasing G, then any of $Z_0(s_0)$ has at most one element. The similar conclusion holds for:*

$$Q(A|i) = \alpha\tilde{Q}(A|i) + (1-\alpha)\mathbf{1}_A(G(i)),$$

whenever $\tilde{Q}(\cdot|i)$ is nonatomic, $\alpha \in (0,1)$ and G is strictly increasing.

The generic set of equilibria is again $\mathcal{G}$ endowed with the weak topology. For any $\mathcal{W} \subset \mathcal{G}$ define

$$\mathcal{B}(\mathcal{W}) := \bigcup_{h \in \mathcal{W}} \arg\max_{c \in [0,s]} P(c, s; h).$$

In other words, $\mathcal{B}$ maps $2^{\mathcal{G}}$ into itself. Obviously $(2^{\mathcal{G}}, \subset)$ is complete lattice and $\mathcal{B}$ is increasing under set inclusion (i.e. $\mathcal{W}_1 \subset \mathcal{W}_2$ implies $\mathcal{B}(\mathcal{W}_1) \subset \mathcal{B}(\mathcal{W}_2)$). By the main theorem in Tarski (1955), there exists a nonempty complete lattice of fixed points (and in particular, the greatest fixed point under set inclusion, i.e. $\mathcal{W}^*$).

Define the sequence of iterations

$$\mathcal{W}^{t+1} = \mathcal{B}(\mathcal{W}^t)$$

for $\mathcal{W}^0 = \mathcal{G}$. Now introduce the following definition:

Definition 9 (Self-generating) *Let $\mathcal{W} \subset \mathcal{G}$. We say that $\mathcal{W}$ is **self-generating** if $\mathcal{W} \subset \mathcal{B}(\mathcal{W})$.*

Denote by $\mathcal{E} \subset \mathcal{G}$ the set of all equilibria. The following lemma shows $\mathcal{E}$ is the greatest self generating set.

Lemma 1 (Self-generating property) *If $\mathcal{W}$ is self generating, then $\mathcal{W} \subset \mathcal{E}$.*

Theorem 6 (Construction of equilibria values) *Let $\mathcal{E}$ to be the set of equilibria. Then, $\mathcal{E}$ is non-empty and it is the greatest fixed point of $\mathcal{B}$. Moreover:*

$$\mathcal{E} = \bigcap_{t=1}^{\infty} \mathcal{W}_t.$$

Proof: Due to series of Lemmas in Balbus et al. (2020), $\mathcal{B}(\mathcal{W})$ is a nonempty weakly compact set whenever $\mathcal{W}$ is. As a result, $\bigcap_{t=1}^{\infty} \mathcal{W}^t \neq \emptyset$ and is weakly compact. Hence: $\mathcal{W}^* \subset \bigcap_{t=1}^{\infty} \mathcal{W}^t$.

On the other hand,

$$\mathcal{B}\left(\bigcap_{t=1}^{\infty}\mathcal{W}^t\right) \subset \mathcal{W}^{t+1},$$

for any t, hence taking intersection over t, we have

$$\mathcal{B}\left(\bigcap_{t=1}^{\infty}\mathcal{W}^t\right) \subset \bigcap_{t=1}^{\infty}\mathcal{W}^t$$

Furthermore,

$$\mathcal{B}\left(\bigcap_{t=1}^{\infty}\mathcal{W}^t\right) \supset \bigcap_{t=1}^{\infty}\mathcal{W}^t.$$

Indeed, $h \in \bigcap_{t=1}^{\infty}\mathcal{W}^t$, hence $h(s) \in \arg\max_{c\in[0,s]} P(c, s; h_t)$ for all s and some sequence h_t such that $h_t \in \mathcal{W}^t$. Since any of $\mathcal{W}^t$ is compact and $\mathcal{W}^t$ is descending, without loss of generality suppose $h_t \to^w \tilde{h}$ for some $\tilde{h} \in \bigcap_{t=1}^{\infty}\mathcal{W}^t$. By Lemma 7 c) in Balbus et al. (2020), $P(c, s; h)$ is continuous in h, hence $h(s) \in \arg\max_{c\in[0,s]} P(c, s; \tilde{h})$, hence $h \in \mathcal{B}(\bigcap_{t=1}^{\infty}\mathcal{W}^t)$. Consequently,

$$\mathcal{W}^* = \bigcap_{t=1}^{\infty}\mathcal{W}^t.$$

Let $\mathcal{E} \subset \mathcal{G}$ be the set of all equilibria continuation values. Observe that $h \in \mathcal{E}$ implies that there is h_t a sequence from $\mathcal{E}$ such that for any $s \in S$, $h(s) \in \arg\max_{c\in[0,s]} P(c, s; h_1)$, $h_1(s) \in \arg\max_{c\in[0,s]} P(c, s; h_2)$, $h_2(s) = \arg\max_{c\in[0,s]} P(c, s; h_3)$ and so on. Consequently $h_1 \in \mathcal{E}$, hence $h \in \mathcal{B}(\mathcal{E})$. In other words $\mathcal{E} \subset \mathcal{B}(\mathcal{E})$, hence $\mathcal{E}$ is self generating.

Obviously any fixed point of $\mathcal{W}$ is self-generating. In particular $\mathcal{W}^*$ is self-generating, hence by Lemma 1 one obtains:

$$\mathcal{W}^* \subset \mathcal{E}. \tag{20}$$

But $\mathcal{E} \subset \mathcal{W}^t$ for any $t \in \mathbb{N}$. hence

$$\mathcal{E} \subset \bigcap_{t=1}^{\infty}\mathcal{W}^t = \mathcal{W}^*.$$

Therefore, by (20),

$$\mathcal{E} = \bigcap_{t=1}^{\infty}\mathcal{W}^t = \mathcal{W}^*.$$

■

CONCLUSION

Existence and characterization of TCPs in a canonical version of dynamic choice problem for a quasi-hyperbolic consumer have been studied extensively in the literature. This chapter presents results on the existence of TCPs in such benchmark model, and also discusses some of its natural generalization. Further, the case of TCPs as SMPE, as well as the case of more general forms of (non-stationary) Markov perfect equilibria via self-generation methods are considered in the chapter. In the case of TCP as SMPE equilibria, we have also given conditions under which uniqueness of TCPs can be established, and we have also discussed when monotone comparative statics of TCP in natural parameters (i.e., discount rates) of the model is possible. Finally, we have discussed when sufficient conditions are present for the existence of generalized Euler equation, and we mention the additional extensions to more general models of dynamic biases.

There is an emerging literature that studies the question of the structure of TCPs in this more general behavioral discounting case. This work includes not only models with present bias, but models with future-bias, backward discounting, hyperbolic discounting models, models with general recursive aggregators that generate dynamically inconsistent preferences, models of altruistic dynastic choice etc. There are also some results on the existence of SMPE in *deterministic* models for the quasi-hyperbolic case (e.g., see Bernheim et al. (2015), Cao and Werning (2018) and Balbus et al. (2022)), as well as work on long-memory solutions for TCPs in models with more general discounting features (see for example Balbus et al. (2021) where many additional results are presented).

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