# FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN ORDERED TOPOLOGICAL SPACES WITH APPLICATIONS TO GAME THEORY.

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**Abstract.** In this paper we prove fixed point theorems for multivalued mappings in ordered topological spaces. Fixed point results in ordered function spaces and in product spaces, and applications to game theory, are also presented.

*Keywords*: Fixed point, multivalued mapping, ordered topological space, ordered metric space, ordered Banach space, ordered function space, product space, generalized iterations, increasing, ascending, meet sublattice, join sublattice, noncooperative game, Nash equilibrium

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## 1. Introduction

In this paper we apply fixed point results derived by a generalized iteration method in [7] to prove fixed point results for multivalued mappings in an ordered topological space $X$. The cases when $X$ is the space of continuous mappings from a separable topological space $Y$ to an ordered Hausdorff space $Y$, when $X$ is the space of vector-valued random variables, and when $X$ is a product space, are also treated. Obtained results are applied to study the existence of Nash equilibria of noncooperative games. Proofs are independent on the Axiom of Choice.

## 2. Preliminaries

By an *ordered topological space* we mean a topological space $X$ equipped with such a partial ordering '$\leq$' that the sets $[a) = \{x \in X \mid a \leq x\}$ and $(b] = \{x \in X \mid x \leq b\}$, and hence also $[a,b] = [a) \cap (b]$, are closed for $a, b \in X$. If the topology of $X$ is induced by a metric, we say that $X$ is an *ordered metric space.* A subset $W$ of an ordered topological space $X$ is called *well-ordered* if each nonempty subset of $W$ has a minimum. In particular, each well-ordered subset of $X$ is a chain. If $Y$ is a partially ordered set (poset), we say

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that a mapping $G\colon Y \to X$ is *increasing* in a subset $W$ of $Y$ if $G(x) \le G(y)$ whenever $x, y \in W$ and $x \le y$ in $Y$.

The following Lemma is a consequence of [7, Theorems 1.1.1 and 1.2.1].

**Lemma 2.1.** *Given $G\colon X \to X$ and $a \in X$, there is a unique well-ordered chain $C$ in $X$, called a well-ordered (w.o.) chain of $G$-iterations of $a$, satisfying*

$$a = \min C, \quad \text{and if } a < x \in X, \text{ then } x \in C \text{ iff } \ x = \sup G[\{y \in C \mid y < x\}]. \tag{2.1}$$

*If $a \le G(a)$, if $G$ is increasing in $[a)$, and if $x_* = \sup G[C]$ exists, then $x_*$ is the smallest fixed point of $G$ in $[a)$ and*

$$x_* = \max C = \min\{y \in [a) \mid G(y) \le y\}. \tag{2.2}$$

For the sake of completeness we state the dual result to Lemma 2.1.

**Lemma 2.2.** *If $G\colon X \to X$ and $b \in X$, there exists exactly one inversely well ordered chain $D$ (each nonempty subset of $D$ has the greatest element) in X, called an i.w.o. chain of $G$-iterations of $b$, such that*

$$b = \max D, \quad \text{and if } X \ni x < b, \text{ then } x \in D \text{ iff } \ x = \inf G[\{y \in D \mid x < y\}]. \tag{2.3}$$

*If $G(b) \le b$, if $G$ is increasing in $(b]$, and if $x^* = \inf G[D]$ exists, then $x^*$ is the greatest fixed point of $G$ in $(b]$, and*

$$x^* = \min D = \max\{y \in (b] \mid y \le G(y)\}. \tag{2.4}$$

**Remark 2.1.** It follows from [7, Lemma 1,1,3] that the least elements of the w.o. chain $C$ of $G$-iterations of $a$ are the elements of the iteration sequences $(G^n a_i)_{n=0}^\infty$ with $a_0 = a$ and $a_{i+1} = \sup_n G^n(a_i)$, $i = 0, 1, \ldots$, as long as these sequences are strictly increasing. In particular, if in Lemma 2.1 $G^n a_i = G^{n+1} a_i$ for some $n$, $i \in \mathbb{N}$, then $x_* = G^n a_i$ is the smallest fixed point of $G$ in $[a)$. Similarly, the greatest elements of the i.w.o. chain $D$ of $G$-iterations of $b$ are the elements of the iteration sequences $(G^n b_i)_{n=0}^\infty$ with $b_0 = b$ and $b_{i+1} = \inf_n G^n(b_i)$, $i = 0, 1, \ldots$, as long as these sequences are strictly decreasing. In particular, if in Lemma 2.2 $G^n b_i = G^{n+1} b_i$ for some $n$, $i \in \mathbb{N}$, then $x^* = G^n b_i$ is the greatest fixed point of $G$ in $(b]$.

## 3. Fixed points results for multivalued mappings

In this section we assume that $X = (X, \le)$ is an ordered topological space which has the following property.

- (X0) Each nonempty well-ordered chain $C$ of $X$ whose increasing sequences converge contains an increasing sequence which converges to $\sup C$, and each nonempty inversely well-ordered chain $D$ of $X$ whose decreasing sequences converge contains a decreasing sequence which converges to $\inf D$.

A nonempty subset $W$ of $X$ is called *downwards closed* if it contains limits of all its decreasing and convergent sequences. If $W$ contains limits of all its increasing and convergent sequences, it is called *upwards closed.* If $W$ is downwards and upwards closed, we say that $W$ is *order closed.* $W$ is called a *meet sublattice* if $\inf\{x,y\}$ exists in $X$ and belongs to $W$ all $x$, $y \in W$. If $\sup\{x,y\}$ exists in $X$ and belongs to $W$ all $x$, $y \in W$, we say that $W$ is a *join sublattice.* A *sublattice* is both meet and join sublattice.

Denote by $2^X$ the set of all subsets of $X$. If $F\colon X \to 2^X \setminus \emptyset$ and $Z$ a nonempty subset of $X$, denote $F[Z] := \cup\{F(x) \mid x \in Z\}$. If $Y$ is a poset, a mapping $F\colon Y \to 2^X \setminus \emptyset$ is called *increasing upwards* if $x \leq \bar{x}$ in $Y$ and $y \in F(x)$ imply that $y \leq \bar{y}$ for some $\bar{y} \in F(\bar{x})$. $F$ is said to be *increasing downwards* if $x \leq \bar{x}$ in $Y$ and $\bar{y} \in F(\bar{t})$ imply the existence of $y \in F(x)$ such that $y \leq \bar{y}$. If $F$ is increasing upwards and downwards, we say that $F$ is *increasing.*

A mapping $F\colon Y \to 2^X \setminus \emptyset$ is called *join ascending* if $\sup\{y, \bar{y}\} \in F(\bar{x})$ whenever $x, \bar{x} \in Y$, $x \leq \bar{x}$, $y \in F(x)$ and $\bar{y} \in F(\bar{x})$. $F$ is *meet ascending* if $\inf\{y,\bar{y}\} \in F(z)$ for all $y \in F(x)$ and $\bar{y} \in F(\bar{x})$ whenever $x$, $\bar{x} \in Y$, $x \leq \bar{x}$. If $F$ is join and meet ascending, we say that $F$ is *ascending* (cf. [12]).

The first fixed point result of this section reads as follows.

**Theorem 3.1.** *Given a nonempty set $T$ and mappings $F^t\colon X \to 2^X \setminus \emptyset$, $t \in T$, assume that the following hypotheses are valid for each $t \in T$.*

(h0) *Monotone sequences of $F^t[X]$ converge.*
(h1) *$F^t$ is increasing downwards, and $F^t(x)$ is a downwards closed and separable meet sublattice of $X$ for each $x \in X$.*
(h2) *There exists an $a^t \in X$ such that $F^t(a^t) \subseteq [a^t)$.*

*Then each $F^t$ has the smallest fixed point $x_t$ in $[a^t)$, and*

$$x_t = \min\{y \in [a^t) \mid \min F^t(y) \leq y\}. \tag{3.1}$$

*The above result holds also when (h1) is replaced by the following hypothesis.*

(h3) *$F^t$ is meet ascending, and $F^t(x)$ is downwards closed and separable for each $x \in X$.*

*Proof.* Assume first that (h0)–(h2) hold. Given $t \in T$ and $x \in X$ we shall first show that $\min F^t(x)$ exists. Since $F^t(x)$ is separable by (h1), it has a countable dense subset $B$. As a countable set $B$ can be represented in the form $B = \{z_n \mid 0 \leq n < m\}$, where $m \in \mathbb{N} \cup \{\infty\}$. Because $F^t(x)$ is a meet sublattice of $X$ by (h1), we can define a sequence $(y_n)$ in $F^t(x)$ by $y_n = \inf\{z_0, \ldots, z_n\}$, $0 \leq n < m$. The so constructed sequence $(y_n)$ is decreasing, whence it has a minimum or a limit $y \in X$ by the hypothesis (h0). Since $F^t(x)$ is downwards closed by (h1), then $y$ belongs to $F^t(x)$. Moreover, $y = \inf_n y_n$ by the dual of [7, Proposition 1.1.3], whence the construction of $(y_n)$ implies that $y$ is a lower bound of $B$. Thus $B$ is a subset of $[y)$. Since $[y)$ is closed, then also $F^t(x)$, as a subset of the closure of $B$, is contained in $[y)$, so that $y = \min F^t(x)$.

The above proof justifies that we can define a mapping $G\colon X \to X$ by

$$G(x) = \min F^t(x), \quad x \in X. \tag{3.2}$$

The hypothesis (h2) implies that $a^t \leq G(a^t)$. $G$ is also increasing in $[a^t)$ because $F^t$ is increasing downwards. Let $C$ be the w.o. chain of $G$-iterations of $a^t$. In view of [9, Corollary 12] we have $G[C] \subseteq C$, so that $G[C]$ is well-ordered and $G[C] \subseteq F^t[C]$. This result, the hypothesis (h0) and property (X0) of $X$ imply that $x_t = \sup G[C]$ exists. It then follows from Lemma 2.1 that $x_t$ is the smallest fixed point of $G$ in $[a^t)$, and (2.2) holds. Thus, by (3.2), $x_t$ is also the smallest fixed point of $F^t$ in $[a^t)$. Relation (3.1) is a direct consequence of (2.2) and (3.2).

By the hypothesis (h3) $F^t$ is meet ascending, which implies that $F^t$ is increasing downwards, and that $F^t(x)$ is a meet sublattice for each $x \in X$. It then follows from (h3) that the hypothesis (h1) holds. □

As an application of Lemma 2.2 we obtain the following dual result to Theorem 3.1.

**Proposition 3.1.** *Given a nonempty set $T$, assume that mappings $F^t\colon X \to 2^X \setminus \emptyset$, $t \in T$, satisfy the hypothesis (h0) and the following hypotheses.*

(h4) *$F^t$ is increasing upwards, and $F^t(x)$ is upwards closed and separable join sublattice for each $x \in X$.*

(h5) *There exists a $b^t \in X$ such that $F^t(b^t) \subseteq (b^t]$.*

*Then $F$ has the greatest fixed point in $(b]$, and*

$$x^t = \max\{y \in (b^t] \mid y \leq \max F^t(y)\}. \tag{3.3}$$

*The above result holds also when (h4) is replaced by the following hypothesis.*

(h6) *$F^t$ is join ascending, and $F^t(x)$ is upwards closed and separable for each $x \in X$.*

The relations (3.1) and (3.3) can be used to prove the following comparison results when $T$ is a poset.

**Proposition 3.2.** *Let $T$ be a poset, and $F^t\colon X \to 2^X \setminus \emptyset$, $t \in T$.*
*a) Assume that the hypotheses of Theorem 3.1 hold, that $t \mapsto F^t(x)$ is increasing downwards for each $x \in X$, and that $t \mapsto a^t$ is increasing. If $x_t$ denotes the smallest fixed point of $F^t$ in $[a^t)$, then $t \mapsto x_t$ is increasing.*
*b) Assume that the hypotheses of Proposition 3.1 hold, that $t \mapsto F^t(x)$ is increasing upwards for each $x \in X$, and that $t \mapsto b^t$ is increasing. If $x^t$ denotes the greatest fixed point of $F^t$ in $(b^t]$, then $t \mapsto x^t$ is increasing.*

*Proof.* a) Let $t, \bar{t} \in T$, $t \leq \bar{t}$, be given. The given hypotheses imply that $\min^t F(x) \leq \min F^{\bar{t}}(x)$ for each $x \in X$. In particular,

$$\min F^t(x_{\bar{t}}) \leq \min F^{\bar{t}}(x_{\bar{t}}) = x_{\bar{t}}.$$

Moreover, $[a_{\bar t} \subseteq [a^t)$, so that $x_{\bar t}$ belongs to the set $\{y \in [a^t) \mid \min F^t(y) \le y\}$. Thus (3.1) implies that $x_t \le x_{\bar t}$.

Applying relation (3.3) a similar reasoning implies the conclusions of case b). □

The existence of elements $a^t$, $b^t \in X$ such that $F^t(a^t) \subseteq [a^t)$ and/or $F^t(b^t) \subseteq (b^t]$ can sometimes be ensured by properties of $X$ defined as follows.

**Definition 3.1.** We say that $c \in X$ in a *sup-center* of $X$ if $\sup\{c, x\}$ exists in $X$ for all $x \in X$, and an *inf-center* of $X$ if $\inf\{c, x\}$ exists in $X$ for all $x \in X$. If $c$ is both sup- and inf-center of $X$, we say that $c$ is an *order center* of $X$.

If $X$ is lattice-ordered, then each point of $X$ is its order center. If $X$ is a meet (respectively join) lattice, then each point of $X$ is its inf- (respectively sup-) center. If $\min X$ (respectively $\max X$) exists, it is a sup-center (respectively an inf-center) of $X$. If $E$ is a Riezs space or a Banach-lattice, then the center of any ball $X$ of $E$ is an order center of $X$. This holds, in particular, if $E$ the Euclidean $m$-space $\mathbb{R}^m$, ordered coordinatewise.

The following result is a consequence of Propositions 3.1 and 3.2.

**Proposition 3.3.** *Given a nonempty set $T$ and mappings $F^t\colon X \to 2^X \setminus \emptyset$, $t \in T$ assume that the hypothesis (h0) and one of the hypotheses (h4) and (h6) hold for each $t \in T$. If $X$ has a sup-center $c$, then each $F^t$ has the greatest fixed point $x^t$ in $(b^t]$, where*

$$b^t = \min\{y \in [c) \mid \sup\{c, \max F^t(y)\} \le y\}. \tag{3.4}$$

*Moreover, if $T$ is a poset, and if the mapping $t \mapsto F^t(x)$ is increasing upwards for each $x \in X$, then the mappings $t \mapsto b^t$ and $t \mapsto x^t$ are increasing.*

*Proof.* Assume that $X$ has a sup-center $c$. Then for each $t \in T$ the relation

$$G(x) = \sup\{c, \max F^t(x)\}, \quad x \in X \tag{3.5}$$

defines an increasing mapping $G\colon X \to X$, and $c \le G(c)$. As in the proof of [10, Theorem 2.1] it can be shown by applying the hypothesis (h0) that if $C$ is the w.o. chain of $G$-iterations of $c$, then $b^t = \sup G[C]$ exists. By Lemma 2.1, $b^t$ is the smallest fixed point of $G$ in $[c)$. It then follows from (2.2) and (3.5) that $b^t$ satisfies (3.4). In particular, $\max F^t(b^t) \le b^t$, so that $F^t(b^t) \subseteq (b^t]$. Thus the hypothesis (h5) holds. Because the other hypotheses of Proposition 3.1 are valid, then $F^t$ has the greatest fixed point $x^t$ in $(b^t]$.

If $T$ is a poset, and if the mapping $t \mapsto F^t(x)$ is increasing upwards for each $x \in X$, it follows from (3.5) that the mapping $t \mapsto b^t$ is increasing. This result and Proposition 3.2 b) imply that also the mapping $t \mapsto x^t$ is increasing. □

As a consequence of Theorem 3.1 and Proposition 3.2 we obtain similarly the following dual result to Proposition 3.3.

**Proposition 3.4.** *Given a nonempty set $T$ assume that mappings $F\colon X \times T \to 2^X \setminus \emptyset$, $t \in T$, satisfy the hypothesis (h0) and one of the hypotheses (h1) and (h3). If $X$ has an inf-center $c$, then each $F^t$ has the smallest fixed point $x_t$ in $[a^t)$, where*

$$a^t = \max\{y \in (c] \mid x \leq \inf\{c, \min F^t(y)\}\}. \tag{3.6}$$

*Moreover, $T$ is a poset, and if the mapping $t \mapsto F^t(x)$ is increasing downwards for each $x \in X$, then the mappings $t \mapsto a^t$ and $t \mapsto x_t$ are increasing.*

As a direct consequence of Propositions 3.3 and 3.4 we get the following result.

**Theorem 3.2.** *Given a nonempty set $T$, assume that mappings $F^t\colon X \to 2^X \setminus \emptyset$, $t \in T$, satisfy the hypothesis (h0) and one of the following hypotheses.*

(h7) *Each $F^t$ is increasing, and their values are order closed and separable sublattices.*
(h8) *Each $F^t$ is ascending, and their values are order closed and separable.*

*If $X$ has an order center $c$, then $F^t$ has for each $t \in T$ smallest and greatest fixed points $x_t$ and $x^t$ in $[a^t, b^t]$, where $b^t$ and $a^t$ are given by (3.4) and (3.6). Moreover, if $T$ is a poset, and if $t \mapsto F^t(x)$ is increasing for each $x \in X$, then all the mappings $t \mapsto x_t$, $t \mapsto a^t$, $t \mapsto x^t$ and $t \mapsto b^t$ are increasing.*

**Remarks 3.1.** The w.o. and i.w.o. chains needed in the proofs are countable by [7, Propositions 1.1.6 and 1.3.6].

All the proofs of above results are independent of Zorn's Lemma and the Axiom of Choice. The separability assumptions given for values of $F$ and $F^t$ can be omitted if the Axiom of Choice is used (cf. [8, Proposition 14]).

In [2] fixed point results are proved for multivalued mappings in ordered topological vector spaces by applying a recursion method presented in [7, Lemma 1.1.1] and the Axiom of Choice.

The following counter-example shows that the result of Theorem 3.1 does not hold, in general, even for a single-valued increasing mapping $F\colon X \to X$ if $X$ does not posses property (X0), or if $X$ is a partially ordered set and topological convergence is replaced by order convergence.

**Example 3.1.** Let $\mathcal{L}^\infty[0,1]$ denote the space of all bounded and Lebesgue-measurable functions $x\colon [0,1] \to \mathbb{R}$. Choose

$$X = \{x \in \mathcal{L}^\infty[0,1] \mid 0 \leq x(t) \leq 1, \quad \text{for all} \quad t \in [0,1]\}.$$

Assuming that $X$ is ordered pointwise, then the pointwise limit of each increasing sequence of $X$ exists and is its supremum in $X$, and the pointwise limit of each decreasing sequence of $X$ exists and is its infimum in $X$. Moreover, the pointwise ordering of $X$ is lattice-ordering. But $X$ is not a complete lattice, whence there exists by [3] an increasing mapping $F : X \to X$ which does not have any fixed point. However, monotone sequences $F[X]$ converge in $X$ with respect to the topology of pointwise convergence, and also with respect to the order convergence of $X$.

## 4. Spaces which have property (X0)

One reason to study fixed point theorems in ordered topological spaces with property (X0) is to simplify the hypotheses. In this section we present examples of topological spaces having property (X0).

- If $X$ satisfies the second countability axiom, then each chain of $X$ is separable, whence property (X0) follows from [7, Lemma 1.1.7] and its dual.
- Each ordered metric space has property (X0) by [7, Proposition 1.1.5] and its dual.
- Each closed subset $X$ of an ordered normed space is an ordered metric space, and thus has property (X0).
- Each weakly sequentially closed subset $X$ of an ordered normed space $E$ satisfies (X0) with respect to the weak topology. This result follows from [1, Lemma A.3.1] and its dual.

Consider next the case when $X$ is a subset of the space $C(Y,Z)$ of continuous functions $x\colon Y \to Z$, where $Y$ is a topological space and $Z$ is an ordered topological space. *In what follows, we assume that $C(X,Y)$, and all its subsets are equipped with the pointwise ordering and the topology of pointwise convergence.*

**Lemma 4.1.** *Let $Y$ be a separable topological space and $Z$ an ordered Hausdorff space which has property (X0). Then each nonempty subset $X$ of $C(Y,Z)$ is an ordered topological space which has property (X0).*

*Proof.* Let $W$ be a well-ordered chain in a nonempty subset $X$ of $C(Y,Z)$, and assume that each increasing sequence of $W$ converges pointwise to a mapping of $X$. For each $s \in Y$ the set $W(s) = \{x(s)\}_{x\in W}$ is a well-ordered chain of $Z$. For if $A$ is a nonempty subset of $W(s)$, then the set $B = \{x \in W \mid x(s) \in A\}$ is nonempty, and thus has the least element $y$. Thus $y(s) = \min A$ because of the pointwise ordering of $C(X,Y)$. Since $Y$ is separable it contains a countable and dense subset $D = \{s_j\}_{j\in\mathbb{N}}$. Let $s \in D$ be fixed, and let $(z_{n_k})_{k=0}^\infty$ be a subsequence of an increasing sequence $(z_n)_{n=0}^\infty$ of $W(s)$. If $(z_{n_k})_{k=m}^\infty$ is a constant sequence for some $m \in \mathbb{N}$, then $z_{n_m}$ is the limit of $(z_{n_k})_{k=0}^\infty$. Otherwise $(z_{n_k})_{k=0}^\infty$ has a strictly increasing subsequence $(z_{n_{k_i}})_{i=0}^\infty$. Since the members of this subsequence belong to $W(s)$, and since $W$ is well-ordered with respect to the pointwise ordering of $C(Y,Z)$, there exists an increasing sequence $(x_i)_{i=0}^\infty$ in $W$ such that $x_i(s) = z_{n_{k_i}}$ for each $i \in \mathbb{N}$ (take $x_i = \min\{x \in W \mid x(s) = z_{n_{k_i}}\}$). Because $(x_i)_{i=0}^\infty$ converges pointwise, then $(x_i(s))_{i=0}^\infty = (z_{n_{k_i}})_{i=0}^\infty$ converges. Consequently, each subsequence of $(z_n)_{n=0}^\infty$ has a convergent subsequence, whence $(z_n)_{n=0}^\infty$ converges by [7, Corollary 1.1.3].

The above proof shows that each increasing sequence $(z_n)_{n=0}^\infty$ of $W(s)$ converges when $s \in D$. Since $W(s)$ is a well-ordered chain in $Z$ which has property (X0), then for each $s_j$, $j \in \mathbb{N}$, there exists an increasing sequence $(x_k^j(s_j))_{k=0}^\infty$ in $W(s_j)$ such that

$$\lim_{k\to\infty} x_k^j(s_j) = \sup W(s_j). \tag{4.1}$$

Denote

$$x_n = \max\{x_k^j \mid 0 \le j,\, k \le n\}, \quad n \in \mathbb{N}. \tag{4.2}$$

The so obtained sequence $(x_n)_{n=0}^\infty$ is and increasing sequence of $W$, whence it converges pointwise to a mapping of $X$ by a hypothesis. Denoting

$$x(s) = \lim_{n\to\infty} x_n(s), \quad s \in Y, \tag{4.3}$$

it follows from (4.1), (4.2) and (4.3) that

$$x(s_j) = \sup W(s_j) \quad \text{for each} \ \ j \in \mathbb{N}. \tag{4.4}$$

To show that $x = \sup W$, let $s \in Y \setminus D$ be given. The above proof shows that there exists an increasing sequence $(y_n)_{n=0}^\infty$ in $W$ such that

$$\lim_{n\to\infty} y_n(s) = \sup W(s). \tag{4.5}$$

Denoting $z_n = \max\{x_n, y_n\}$, $n \in \mathbb{N}$, we obtain an increasing sequence $(z_n)_{n=0}^\infty$ of $W$. Denoting by $z$ its limit function, which is continuous by a hypothesis, it follows from (4.4) and (4.5) that

$$z(s_j) = \sup W(s_j), \ j \in \mathbb{N} \text{ and } z(s) = \sup W(s). \tag{4.6}$$

Both $x$ and $z$ are continuous, their restrictions to the dense subset $D$ of $Y$ are equal by (4.4) and (4.6). Since $Z$ is a Hausdorff space, then $z = x$. In particular,

$$x(s) = \lim_{n\to\infty} x_n(s) = \sup W(s).$$

This result holds for all $s \in Y \setminus D$. It holds by (4.3) and (4.4) also for all $s \in D$, whence $x$ is a pointwise supremum of $W$. Obviously, $x = \sup W$ with respect to the pointwise ordering of $X$. Moreover, $x$ is a pointwise limit of an increasing sequence $(x_n)_{n=0}^\infty$ of $W$.

The proof that each inversely well-ordered chain $W$ of $X$ whose decreasing sequences converge pointwise in $X$ contains a decreasing sequence which converges pointwise to $\inf W$ in $X$ is dual to the above proof. □

Assume next that $Y$ is a topological space and $Z = (Z, d)$ is a metric space. We say that a subset $W$ of $C(Y, Z)$ is *equicontinuous* if for each $t \in Y$ and for each $\epsilon > 0$ there exists a neighbourhood $U$ of $t$ such that

$$d(x(s), x(t)) \le \epsilon \ \text{ for all } x \in W \text{ and } s \in U.$$

The next result is an easy consequence of the proof of [7, Proposition 1.3.8].

**Lemma 4.2.** *Let $Y$ be a topological space and $Z = (Z, d)$ an ordered metric space. If a pointwise monotone and equicontinuous sequence of functions from $Y$ to $Z$ has a pointwise limit, this limit function is continuous.*

As a consequence of Lemma 4.1, Lemma 4.2 we obtain the following result.

**Corollary 4.1.** *Assume that $Y$ is a separable topological space, and that $Z$ is an ordered metric space. Then the fixed point results of Section 3 hold when $X$ is a nonempty closed subset of $C(Y,Z)$, if the hypothesis (h0) is replaced by*

(h) *Monotone sequences of $F^t[X]$ are equicontinuous and converge pointwise for each $t \in T$.*

*Proof.* Since $Z$ is an ordered metric space it is a Hausdorff space and it has property (X0). It then follows from Lemma 4.1 that $X$ has also this property. If $(x_n)_{n=0}^{\infty}$ is a monotone sequence in $F^t[X]$, $t \in T$, it is equicontinuous and converges pointwise to a function $x\colon Y \to Z$ by a hypothesis. Lemma 4.2 implies that $x$ is continuous. Because $F^t[X] \subseteq X$, and $X$ is closed, then $x \in X$.

The above proof shows that monotone sequences of $F^t[X]$ converge for each $t \in T$, whence the hypothesis (h0) holds. □

If $Y$ is compact metric space, it is also separable. Moreover, if $Z$ is an ordered metric space, the functions of $C(Y,Z)$ are uniformly continuous, and we can define a metric $\rho$ in $C(Y,Z)$ by $\rho(x,y) = \sup\{d(x(t),y(t)) \mid t \in Y\}$. The proof which is similar to that given in [4, Section 7.5] in the case when $Z$ is a Banach space, shows that if an equicontinuous sequence of $C(Y,Z)$ converges pointwise, it converges in $(C(Y,Z),\rho)$.

Let $B$ be a closed and bounded ball in a separable and weakly sequentially complete ordered Banach space $E$ whose order cone is normal. Denote by $\mathcal{B}$ the $\sigma$-algebra of Borel sets of $B$. Let $(\Omega,P)$ denote a probability space and $X$ the space of all $B$-valued random variables on $(\Omega,P)$, i.e. measurable mappings $x\colon(\Omega,P) \to (B,\mathcal{B})$. Define a *Ky Fan* metric $\alpha$ and a partial ordering $\leq_r$ in $X$ by

$$\begin{cases} \alpha(x,y) = \inf\{\epsilon > 0 \mid P\{\omega \in \Omega \mid (\|x(\omega) - y(\omega)\| > \epsilon\} \leq \epsilon\}, \\ x \leq_r y \ \text{ if and only if } \ P\{\omega \in \Omega \mid x(\omega) \leq y(\omega)\} = 1. \end{cases}$$

It can be shown that $(X,\leq_r,\alpha)$ is an ordered metric space.

**Lemma 4.3.** *Each monotone sequence of $(X,\leq_r)$ converges in $(X,\alpha)$.*

*Proof.* Let $(x_n)_{n=1}^{\infty}$ be a monotone sequence in $(X,\leq_r)$. The definition of $\leq_r$ implies that for almost all $\omega \in \Omega$ the sequence $(x_n(\omega))_{n=1}^{\infty}$ of $E$ is monotone. Since $B$ is bounded, it follows from [6, Theorem 2.4.5] that the limit $x(\omega) = \lim_{n\to\infty} x_n(\omega)$ exists for almost all $\omega \in \Omega$. Thus $x$ is (equal a.s. to) a $B$-valued random variable by [5, Theorem 4.2.2], and $x_n \to x$ a.s., and hence also in probability. Because the Ky Fan metric $\alpha$ metrizes the convergence in probability by [5, Theorem 9.2.2], then $\alpha(x_n,x) \to 0$. □

The next result is a consequence of Theorems 3.1, 3.2 and 3.3 and Lemma 4.3.

**Corollary 4.2.** *Let $B$ be a bounded ball in a separable and weakly sequentially complete ordered Banach space whose order cone is normal, and let $X$ be the space of all $B$-valued random variables on a probability space $(\Omega,P)$, metrized by the Ky Fan metric. Then the hypothesis (h0) of Theorem 3.1 holds for all mappings $F\colon X \times T \to 2^X \setminus \emptyset$.*

## 5. Fixed point results in product spaces

In this section we consider the case when $X$ is a product space: $X = \prod_{i\in I} X_i$, where $I$ is an index set and each $X_i = (X_i, \le_i)$ is an ordered topological space. Define a partial ordering $\le$ on $X$ by

$$\{x_i\}_{i\in I} \le \{y_i\}_{i\in I} \ \text{ iff } x_i \le_i y_i \text{ for each } \ i \in I. \tag{5.1}$$

Denoting by $p_j$ the projection mapping from $X$ onto $X_j$, $j \in I$, it follows from (5.1) that a sequence $(y_n)_{n=0}^\infty$ of $X$ is increasing (respectively decreasing) iff $p_j(y_n) \le_j p_j(y_{n+1})$ (respectively $p_j(y_{n+1}) \le_j p_j(y_n)$) for all $j \in I$ and $n \in \mathbb{N}$.

**Lemma 5.1.** *Assume that each coordinate space $X_i = (X_i, \le_i)$ of a product space $X = \prod_{i\in I} X_i$ has property (X0).*
*a) If $C$ is a well-ordered chain in $X$, and if increasing sequences of $p_j(C)$ converge in $X_j$ for each $j \in I$, then $\sup C$ exists.*
*b) If $D$ is an inversely well-ordered chain in $X$, and if decreasing sequences of $p_j(D)$ converge in $X_j$ for each $j \in I$, then $\inf D$ exists.*

*Proof.* a) Let $C$ be a well-ordered chain in $X$, and assume that each increasing sequence of $p_j(C)$ converges in $X_j$ for each $j \in I$. To show that each projection $p_j(C)$ of $C$ is a well-ordered subset of $(X_j, \le_j)$, let $j \in I$ be fixed, and let $A$ be a nonempty subset of $p_j(C)$. Then the set $B = \{x \in C \mid p_j(x) \in A\}$ is a nonempty subset of $C$, whence $y = \min B$ exists. This result and (5.1) imply that $p_j(y) = \min A$. Since $p_j(C)$ is a well-ordered chain in $X_j$ which has property (X0), and since each increasing sequence of $p_j(C)$ converges in $X_j$, then $x_j = \sup p_j(C)$ exists in $X_j$, and it is a limit of an increasing sequence of $p_j(C)$. This result holds for each $j \in I$, whence the definition (5.1) of the partial ordering of $X$ implies that $x = \{x_i\}_{i\in I}$ is the supremum of $C$ in $X$.

The proof of case b) is dual to the above one. □

With the help of Lemma 5.1 we are now able to generalize the results of Section 3 to the case when $X$ is a product of ordered topological spaces having property (X0). For instance, Theorem 3.1 can be generalized as follows.

**Theorem 5.1.** *Let $T$ be a nonempty set, and let each coordinate space $X_i = (X_i, \le_i)$ of $X = \prod_{i\in I} X_i$ posses property (X0). Assume that each mapping $F_i^t \colon X \to 2^{X_i} \setminus \emptyset$, $t \in T$, $i \in I$, satisfies the following hypotheses.*

- (hi0) *Monotone sequences of $F_i^t[X]$ converge in $X_i$.*
- (hi1) *$F_i^t$ is increasing downwards, and $F_i^t(x)$ is a downwards closed and separable meet sublattice of $X_i$ for all fixed $x \in X$.*
- (hi2) *There exists an $a^t = \{a_i^t\}_{i\in I} \in X$ such that $F_i^t(a^t) \subseteq [a_i^t)$.*

*Then for each $t \in T$ the mapping $F^t = \{F_i^t\}_{i\in I}$ has the smallest fixed point $x_t$ in $[a^t)$, and*

$$x_t = \min\{y \in [a^t) \mid \min F^t(y) \le y\}. \tag{5.2}$$

*If $T$ is a poset, if $t \mapsto F_i^t(x)$ is increasing downwards for each $i \in I$ and $x \in X$, and if $t \mapsto a_i^t$ is increasing for each $i \in I$, then the mapping $t \mapsto x_t$ is increasing.*
*The above results hold also when (hi1) is replaced by the following hypothesis.*

(hi3) *Each $F_i^t$ is meet ascending, and $F_i^t(x)$ is a downwards closed and separable subset of $X_i$ for each $x \in X$.*

*Proof.* Applying the hypothesis (hi0) one can show as in the proof of Theorem 3.1 that $\min F_i^t(x)$ exists for all fixed $i \in I$, $t \in T$ and $x \in X$ so that, by (5.1), $F^t(x) = \{F_i^t(x)\}_{i \in I}$ has the minimum for all $t \in T$ and $x \in X$. Thus, for each fixed $t \in T$ we can define a mapping $G_t \colon X \to X$ by

$$G_t(x) = \min F^t(x) = \{\min F_i^t(x)\}_{i \in I}, \quad x \in X. \tag{5.3}$$

$G_t$ is increasing because $F^t$ is by (hi1) and (5.1) increasing. Let $C$ be the w.o. chain of $G_t$-iterations of $a^t$. Because $C$ is well-ordered and $G_t$ is increasing, then $G_t[C]$ is well-ordered and $G_t[C] \subseteq F^t[C]$. This result, the hypothesis (hi0) and Lemma 5.1 imply that $x_t = \sup G_t[C]$ exists. It then follows from Lemma 2.1 that $x_t$ is the smallest fixed point of $G_t$ in $[a^t)$, and

$$x_t = \min\{y \in [a^t) \mid G_t(y) \le y\}.$$

Thus, by (5.3), $x_t$ is also the smallest fixed point of $F^t$ in $[a^t)$, and (5.2) holds.

The above result holds for each $t \in T$. If $T$ is a poset, and if $t \mapsto F_i^t(x)$ is increasing downwards for each $i \in I$ and $x \in X$, then the mapping $t \mapsto \min F^t(x) = \{\min F_i^t(x)\}_{i \in I}$ is increasing for each $x \in X$. Moreover, if $t \mapsto a_i^t$ is increasing for each $i \in I$, then the mapping $t \mapsto a^t = \{a_i^t\}_{i \in I}$ is increasing. Thus the relation (5.2) can be used, as in the proof of Proposition 3.2, to show that the mapping $t \mapsto x_t$ is increasing.

If the hypothesis (hi3) holds, then each $F_i^t$ is increasing downwards, and $F_i^t(x)$ is a meet sublattice of $X_i$ for all fixed $i \in I$, $x \in X$ and $t \in T$. Thus (hi1) holds. □

As a dual result of Theorem 5.1 one can prove similarly the following generalization to Proposition 3.1, by applying Lemma 2.2 instead of Lemma 2.1.

**Proposition 5.1.** *Let $T$ and $X = \prod_{i \in I} X_i$ be as in Theorem 5.1. Assume that each mapping $F_i^t \colon X \to 2^{X_i} \setminus \emptyset$, $t \in T$, $i \in I$, satisfies (hi0) and the following hypotheses.*

(hi4) *$F_i^t$ is increasing upwards in $X$, and $F_i^t(x)$ is an upwards closed and separable join sublattice of $X_i$ for all fixed $x \in X$.*

(hi5) *There exists a $b^t = \{b_i^t\}_{i \in I} \in X$ such that $F_i^t(b^t) \subseteq (b_i^t]$.*

*Then for each $t \in T$ the mapping $F^t = \{F_i^t\}_{i \in I}$ has the greatest fixed point $x^t$ in $(b^t]$, and*

$$x^t = \max\{y \in (b^t] \mid y \le \max F^t(y)\}. \tag{5.4}$$

*If $T$ is a poset, if $t \mapsto F_i^t(x)$ is increasing upwards for each $i \in I$ and $x \in X$, and if $t \mapsto b_i^t$ is increasing for each $i \in I$, then the mapping $t \mapsto x^t$ is increasing.*
*The above results hold also when (hi4) is replaced by the following hypothesis.*

(hi6) *Each $F_i^t$ is join ascending in $X$, and $F_i^t(x)$ is an upwards closed and separable subset of $X_i$ for each $x \in X$.*

The following result is a generalization to Proposition 3.3.

**Proposition 5.2.** *Let $T$ and $X = \prod_{i\in I} X_i$ be as in Theorem 5.1. Assume that mappings $F_i^t \colon X \to 2^{X_i} \setminus \emptyset$, $t \in T$, $i \in I$, satisfy the hypothesis (hi0) and one of the hypotheses (hi4) and (hi6). If each $X_i$ has a sup-center $c_i$, then each $F^t = \{F_i^t\}_{i\in I}$ has the greatest fixed point $x^t$ in $(b^t]$, where*

$$b^t = \min\{y \in [c) \mid \sup\{c, \max F^t(y)\} \le y\}, \quad c = \{c_i\}_{i\in I}. \tag{5.5}$$

*Moreover, if $T$ is a poset, and if the mapping $t \mapsto F_i^t(x)$ is increasing upwards for each $i \in I$ and $x \in X$, then the mappings $t \mapsto b^t$ and $t \mapsto x^t$ are increasing.*

The next result is a generalization to Proposition 3.4.

**Proposition 5.3.** *Let $T$ and $X = \prod_{i\in I} X_i$ be as in Theorem 5.1, and assume that mappings $F_i^t \colon X \to 2^{X_i} \setminus \emptyset$, $t \in T$, $i \in I$, satisfy the hypothesis (hi0) and one of the hypotheses (hi1) and (hi3). If each $X_i$ has an inf-center $c_i$, then each $F^t = \{F_i^t\}_{i\in I}$ has the smallest fixed point $x_t$ in $[a^t)$, where*

$$a^t = \max\{y \in (c] \mid x \le \inf\{c, \min F^t(y)\}\}, \quad c = \{c_i\}_{i\in I}. \tag{5.6}$$

*Moreover, if $T$ is a poset, and if the mapping $t \mapsto F_i^t(x)$ is increasing downwards for each $i \in I$ and $x \in X$, then the mappings $t \mapsto a^t$ and $t \mapsto x_t$ are increasing.*

As a direct consequence of Propositions 5.2 and 5.3 we get the following result.

**Theorem 5.2.** *Let $T$ and $X = \prod_{i\in I} X_i$ be as in Theorem 5.1. Assume that mappings $F_i^t \colon X \to 2^{X_i} \setminus \emptyset$, $t \in T$, satisfy the hypothesis (hi0) and one of the following hypotheses.*

- (hi7) *Each $F_i^t$ is increasing, and their values are order closed and separable sublattices.*
- (hi8) *Each $F_i^t$ is ascending, and their values are order closed and separable.*

*If each $X_i$ has an order center $c_i$, then $F^t = \{F_i^t\}_{i\in I}$ has for each $t \in T$ smallest and greatest fixed points $x_t$ and $x^t$ in $[a^t, b^t]$, where $b^t$ and $a^t$ are given by (5.5) and (5.6). Moreover, if $T$ is a poset, and if $t \mapsto F^t(x)$ is increasing for each $x \in X$, then all the mappings $t \mapsto x_t$, $t \mapsto a^t$, $t \mapsto x^t$ and $t \mapsto b^t$ are increasing.*

**Remarks 5.1.** In the case when each $X_i$ is a nonempty closed subset of an ordered normed space $E_i$, the hypothesis (hi0) can be replaced by one of the following hypotheses.

- (i) The order cone of $E_i$ is regular or fully regular, and $F_i^t[X]$ is order-bounded.
- (ii) The order cone of $E_i$ is fully regular, and $F_i^t[X]$ is order- or norm-bounded.
- (iii) $E_i$ is reflexive, $X_i$ is equipped with weak topology, and $F_i^t[X]$ is norm-bounded.
- (iv) $E_i$ is weakly sequentially complete, the order cone of $E_i$ is normal, and $F_i^t[X]$ is order- or norm-bounded.

Each of the following spaces equipped with a $p$-norm are ordered Banach spaces with fully regular order cone when $1 \le p < \infty$.

a) $\mathbb{R}^m$, ordered coordinatewise.
b) $l^p$, ordered componentwise.
c) $L^p(\Omega, E)$, ordered a.e. pointwise, where $\Omega = (\Omega, \mathcal{A}, \mu)$ is a measure space and $E$ is an ordered Banach space with fully regular order cone (cf. [7, Proposition 5.8.7]).

As for proofs of the above properties and further examples of ordered Banach spaces $E$ whose order cones are regular or fully regular, see, e.g., [1,6,11].

The spaces given in a)–c) above are also reflexive if $1 < p < \infty$, $\Omega$ is a domain in $\mathbb{R}^m$ and $E = \mathbb{R}$. Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$, $k \in \mathbb{N}$, ordered a.e. pointwise, are reflexive ordered Banach spaces. Moreover, these spaces and those given in a) and b) are lattice-ordered, whence each of their points are their order centers.

## 6. Applications to game theory

We shall now apply results of Section 5 to study the existence of Nash equilibria of multi-firm noncooperative games. Following the terminology adopted in [12], let $I$ be a set of firms and $T$ be a poset of exogenous parameters reflecting the environment in which the firms compete. Let $X := \Pi_{i\in I} X_i$ be the product of *strategy posets* $X_i$ of the firm s and $X^i := \Pi_{j\in I\setminus\{i\}} X_j$ be the product of strategies of firms other than $i$. Given a strategy $s = \{s_i\}_{i\in I} \in X$ of the firms and $i \in I$, we use the notation $s = (s^i, s_i)$, where $s_i \in X_i$ denotes firm $i$'s strategy and $s^i \in X^i$ the strategy of other firms.

Let $X_i^{s,t} \subseteq X_i$ be firm $i$'s *feasible replies* to $(s,t)$, and let $u_i^t(s)$ denote a poset $Z_i$-valued *utility* to firm $i \in I$ when $s \in X$ is a strategy of the firms and $t \in T$ is an exogenous parameter. Let $F_i^t(s) \subseteq X_i^{s,t}$ be firm $i$'s *optimal replies* to $(s,t) \in X \times T$, i.e., each point of $F_i^t(s)$ maximizes the values of the function $u_i^{s,t} \colon X_i^{s,t} \to Z_i$, defined by

$$u_i^{s,t}(\sigma) := u_i^t(s^i, \sigma), \quad \sigma \in X_i^{s,t}. \tag{6.1}$$

Call the strategy $s \in X$ a *Nash equilibrium* for $t \in T$ if $s \in F^t(s) = \Pi_{i\in I} F_i^t(s)$.

In what follows we assume that

(A) *$X_i$:s and $Z_i$:s are ordered second countable topological spaces.*

Our first existence result for Nash equilibria is a consequence of Proposition 5.2.

**Proposition 6.1.** *Assume that the following hypotheses are valid.*

(H0) *Monotone sequences of* $\cup\{X_i^{s,t} \mid s \in X\}$ *converge in* $X_i$ *for all* $i \in I$ *and* $t \in T$.
(H1) *Each* $X_i^{s,t}$ *is upwards closed.*
(H2) *If* $(s,t) \leq (\overline{s}, \overline{t})$ *in* $X \times T$, $i \in I$, $\sigma \in X_i^{s,t}$ *and* $\tau \in X_i^{\overline{s},\overline{t}}$, *then* $\sup\{\sigma, \tau\} \in X_i^{\overline{s},\overline{t}}$, *and* $u_i^{s,t}(\sigma) \leq u_i^{\overline{s},\overline{t}}(\sup\{\sigma,\tau\})$.

*If each $X_i$ is a join lattice, then for each $t \in T$ and for each choice of $c_i \in X_i$ there exist the greatest Nash equilibrium $s^t$ for $t$ in the order interval $(b^t]$ of $X$, where $b^t$ is given by (5.5). Moreover, the mappings $t \mapsto b^t$ and $t \mapsto s^t$ are increasing.*

*Proof.* We shall first show that each $F_i^t(s)$ is nonempty. For fixed $i \in I$, $s \in X$ and $t \in T$ the set $u_i^{s,t}[X_i^{s,t}]$ is separable by (A), whence it has a countable dense subset

$B = \{u_i^{s,t}(\sigma_n) \mid 0 \le n < m\}$, where $1 \le m \le \infty$. It follows from (H2) that denoting $\tau_n = \sup\{\sigma_0, \dots, \sigma_n\}$ and $y_n = u_i^{s,t}(\tau_n)$, $0 \le n < m$, one obtains increasing sequences $(\tau_n)$ in $X_i^{s,t}$ and $(y_n)$ in $u_i^{s,t}[X_i^{s,t}]$. In view of (H0), $(\tau_n)$ has a maximum or a limit $\tau \in X_i$. Since $X_i^{s,t}$ is upwards closed by (H1), then $\tau$ belongs to $X_i^{s,t}$. Moreover, $\tau = \sup_n \tau_n$ by [7, Proposition 1.1.3]. Denoting $y = u_i^{s,t}(\tau)$, the hypothesis (H2) and the definitions of $\tau_n$ and $y_n$ imply that

$$u_i^{s,t}(\sigma_n) \le u_i^{s,t}(\tau_n) = y_n \le y$$

for each $n$. Thus $B$ is a subset of $(y]$. Since $(y]$ is closed, then also $u_i^{s,t}[X_i^{s,t}]$, as a subset of the closure of $B$, is contained in $(y]$, so that $y = \max u_i^{s,t}[X_i^{s,t}]$, i.e., $y \in F_i^t(s)$.

Each $F_i^t(s)$ is separable by (A). To prove that $F_i^t(s)$ is upwards closed, let $(\tau_n)$ be an increasing sequence of $F_i^t(s)$ which converges in $X_i$. Since $F_i^t(s) \subseteq X_i^{s,t}$, it follows from (H1) that $\tau = \lim_n \tau_n \in X_i^{s,t}$. Moreover, $u_i^{s,t}(\tau) \in F_i^t(s)$, since the hypothesis (H2) implies that $y = \max u_i^{s,t}[X_i^{s,t}] = u_i^{s,t}(\tau_1) \le u_i^{s,t}(\sup \tau_1, \tau) = u_i^{s,t}(\tau)$.

The hypothesis (H0) implies that the hypothesis (hi0) of Theorem 5.1 holds. The hypothesis (H2) ensures that each $F_i^t(s)$ is a join sublattice, and that each mapping $(s,t) \mapsto F_i^t(s)$ is increasing upwards. In particular, the mapping $F_i^t$ is increasing upwards and its values are separable and upwards closed join sublattices for all $t \in T$ and $i \in I$, whence the hypothesis (hi4) of Proposition 5.1 holds. If each $X_i$ is a join lattice, then for each choice of $c_i \in X_i$, $i \in I$, and for each $t \in T$ the mapping $F^t = \{F_i^t\}_{i \in I}$ has by Proposition 5.2 the greatest fixed point $s^t$ in $(b^t]$, where $b^t$ is given by (5.5). By definition, $s^t$ is also the greatest Nash equilibrium for $t$ in $(b^t]$. Because the mapping $t \mapsto F^t(s)$ is increasing for each $s \in X$, the last assertion follows from last conclusion of Proposition 5.2. □

In the next consequence of Proposition 5.1 we give sufficient conditions for the existence of such a Nash equilibrium which maximizes utilities of each firm.

**Proposition 6.2.** *Assume that the hypotheses (H0)–(H2) of Proposition 6.1 hold, and that*

(H3) $\cup\{X_i^{s,t} \mid s \in X\}$ *is bounded from above for all* $i \in I$ *and* $t \in T$.

*Then the greatest Nash equilibrium* $\overline{s}_t$ *exists for every* $t \in T$. *Moreover, each* $\overline{s}_t$ *maximizes each utility* $u_i^t(s)$ *over all Nash equilibria for* $t$ *if*

(H4) *the mapping* $s^i \mapsto u_i^t(s^i, s_i)$ *is increasing for all* $i \in I$, $s_i \in X_i$, $t \in T$.

*Proof.* The hypotheses (H0)–(H2) imply by the proof of Proposition 6.1 that the hypotheses (hi0) and (hi4) of Proposition 5.1 hold. In view of (H3) there exist $b_i^t \in X_i$ such that $F_i^t[X] \subseteq (b_i^t]$ for all $i \in I$ and $t \in T$. This result implies that also the hypothesis (hi5) of Proposition 5.1 is satisfied. Thus the mapping $F^t$ has for each $t \in T$ the greatest fixed point $\overline{s}_t$ in $(b^t]$, where $b^t = \{b_i^t\}_{i \in I}$. Since $F^t[X] \subseteq (b^t]$ for each $t \in T$, then each $\overline{s}_t$ is the greatest of all fixed points of $F^t$, and hence the greatest of all Nash equilibria for $t$. This proves the first assertion. This result the hypothesis (H4) and [12, Theorem 3 of Chapter 10] implies second assertion. □

The following dual result to Proposition 6.1 follows from Proposition 5.3.

**Proposition 6.3.** *Assume that the hypothesis (H0) of Proposition 6.1 and the following hypotheses are valid.*

(H5) *Each $X_i^{s,t}$ is downwards closed.*

(H6) *If $(s,t) \le (\overline{s},\overline{t})$ in $X \times T$, $i \in I$, $\sigma \in X_i^{s,t}$ and $\tau \in X_i^{\overline{s},\overline{t}}$, then $\inf\{\sigma,\tau\} \in X_i^{s,t}$, and $u_i^{s,t}(\inf\{\sigma,\tau\}) \ge u_i^{\overline{s},\overline{t}}(\tau)$.*

*If each $X_i$ is a meet lattice, then for each $t \in T$ and for each choice of $c_i \in X_i$ there exist the smallest Nash equilibrium $s_t$ for $t$ in the order interval $[a^t)$ of $X$, where $a^t$ is given by (5.6). Moreover, the mappings $t \mapsto a^t$ and $t \mapsto s_t$ are increasing.*

The next result is a consequence of Theorem 5.1.

**Proposition 6.4.** *Assume that the hypotheses (H0), (H5) and (H6) of Proposition 6.3 hold, and that*

(H7) *$\cup\{X_i^{s,t} \mid s \in X\}$ is bounded from below for all $i \in I$ and $t \in T$.*

*Then the smallest Nash equilibrium $\underline{s}_t$ exists for every $t \in T$. Moreover, each $\underline{s}_t$ maximizes each utility $u_i^t(s)$ over all Nash equilibria for $t$ if*

(H8) *the mapping $s^i \mapsto u_i^t(s^i, s_i)$ is decreasing for all $i \in I$, $s_i \in X_i$, $t \in T$.*

The following Lemma offers alternatives to hypotheses (H1) and (H5).

**Lemma 6.1.** a) *The results of Propositions 6.1 and 6.2 hold if the hypothesis (H1) is replaced by the following hypothesis*

(H9) *Each $u_i^{s,t}[X_i^{s,t}]$ is closed upwards, its increasing sequences converge, and for each $y \in u_i^{s,t}[X_i^{s,t}]$ the set $\{\sigma \in X_i^{s,t} \mid u_i^{s,t}(\sigma) = y\}$ is closed upwards.*

*b) The results of Propositions 6.3 and 6.4 hold if the hypothesis (H5) is replaced by the following hypothesis*

(H10) *Each $u_i^{s,t}[X_i^{s,t}]$ is closed downwards, its decreasing sequences converge, and for each $y \in u_i^{s,t}[X_i^{s,t}]$ the set $\{\sigma \in X_i^{s,t} \mid u_i^{s,t}(\sigma) = y\}$ is closed downwards.*

*Proof.* a) Let $i \in I$, $t \in T$ and $s \in X$ be given. It suffices to show that $F_i^t(s)$ is nonempty and upwards closed. $B = \{u_i^{s,t}(\sigma_n) \mid 0 \le n < m\}$, $1 \le m \le \infty$, be a countable dense subset of $u_i^{s,t}[X_i^{s,t}]$, and let $(y_n)$ be an increasing sequence of $u_i^{s,t}[X_i^{s,t}]$ constructed in the proof of Proposition 6.1. The hypothesis (H9) and [7, Proposition 1.1.3] imply that $y = \sup_n y_n$ exists in $Z_i$ and belongs to $u_i^{s,t}[X_i^{s,t}]$. In view of the construction of $(y_n)$ we see that $B$ is contained in $(y]$, which is closed, whence $u_i^{s,t}[X_i^{s,t}] \subseteq \overline{B} \subseteq (y]$. Thus $y = \max u_i^{s,t}[X_i^{s,t}]$, so that $F_i^t(s)$ is nonempty. The last condition of (H9) implies that $F_i^t(s)$ is closed upwards.

The proof of b) is similar. $\square$

In the next Lemma we list sufficient conditions for the validity of some hypotheses used above.

**Lemma 6.2.** *a) (A) holds if $X_i$:s and $Z_i$:s are separable metric spaces.*
*b) (H2) and (H4) hold if $(s,t) \mapsto X_i^{s,t}$ is join ascending and $(s,t) \mapsto u_i^t(s)$ is increasing for each $i \in I$.*
*c) (H6) and (H8) hold if $(s,t) \mapsto X_i^{s,t}$ is meet ascending and $(s,t) \mapsto u_i^t(s)$ is decreasing for each $i \in I$.*
*d) (A), (H0), (H3) and (H7) hold if $X_i$ is a separable normed space with regular or fully regular order cone and $\cup\{X_i^{s,t} \mid s \in X\}$ is order-bounded for all $i \in I$ and $t \in T$.*
*e) (A) and (H0) hold if $X_i$ is a separable normed space with fully regular order cone and $\cup\{X_i^{s,t} \mid s \in X\}$ is norm-bounded for all $i \in I$ and $t \in T$.*

**Remarks 6.1.** Each of the spaces given in Remarks 5.1. a) and b) is a separable normed space with fully regular order cone. In view of Lemma 4.3 all the monotone sequences of the ordered metric space of vector-valued random variables constructed in Section 4 converge. It can also be shown that this space is separable.

According to Remark 2.1 it follows from the proofs of the fixed point results of Section 5, that if the hypotheses of Proposition 6.1 hold, and if for some $t \in T$ strictly monotone sequences of $\{\max F^t(s) \mid s \in X\}$ are finite, then $b^t$ given by (5.5), and the greatest Nash equilibrium $s^t$ for $t$ in the order interval $(b^t]$ can be computed as follows.

$b^t$ is the last element of the finite sequence of iterations

$b_0 = c, \quad \text{and} \quad b_{n+1} = \sup\{c, \max F^t(b_n)\}, \; c = \{c_i\}_{i\in I}$ as long as $b_n < b_{n+1}$.

$s^t$ is the last element of the finite sequence of iterations

$y_0 = b^t, \quad \text{and} \quad y_{n+1} = \max F^t(y_n), \;$ as long as $y_{n+1} < y_n$.

If the hypotheses of Proposition 6.2 hold, and if strictly decreasing sequences of $\{\max F^t(s) \mid s \in X\}$ are finite for some $t \in T$, the greatest Nash equilibrium $\overline{s}_t$ for $t$ is the last element of iterations:

$y_0 = b^t, \quad \text{and} \quad y_{n+1} = \max F^t(y_n), \;$ as long as $y_{n+1} < y_n$.

If the hypotheses of Proposition 6.3 hold, and if strictly monotone sequences of $\{\min F^t(s) \mid s \in X\}$ are finite for some $t \in T$, then $a^t$ given by (5.6) and the smallest Nash equilibrium $s_t$ for $t$ in $[a^t)$ can be calculated as follows.

$a^t$ is the last element of the finite sequence of iterations

$a_0 = c, \quad \text{and} \quad a_{n+1} = \inf\{c, \min F^t(a_n)\}, \; c \in S, \;$ as long as $a_{n+1} < a_n$.

$s_t$ is the last element of the finite sequence of iterations

$x_0 = a^t, \quad \text{and} \quad x_{n+1} = \min F^t(x_n), \;$ as long as $x_n < x_{n+1}$.

If the hypotheses of Proposition 6.4 hold, and if for some $t \in T$ strictly increasing sequences of $\{\min F^t(s) \mid s \in X\}$ are finite, the smallest Nash equilibrium $\underline{s}_t$ for $t$ is the last element of iterations:

$x_0 = a^t, \quad \text{and} \quad x_{n+1} = \min F^t(x_n), \;$ as long as $x_n < x_{n+1}$.

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