Iterative Monotone Comparative Statics*

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Abstract

We propose a novel approach to equilibrium comparative statics in economic environments where complementarities play a critical role, including environments in which the existing methods for obtaining monotone comparative statics appear inadequate. Our approach is dynamic and, methodologically, in the spirit of the celebrated "correspondence principle", a concept first presented in the work of Samuelson (1947). It applies even to: (a) environments with a continuum of equilibria (b) environments in which all equilibria are unstable; and (c) chaotic environments, in which adaptive learning sequences may not converge. Finally, our propositions extend several previously existing results.

Keywords: comparative statics; adaptive learning; monotone iterations; economic environments with complementarities

JEL classification: C62, C65, C72

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1 Introduction

Comparative statics has always been a foundational tool of economic analysis. It asks how the set of optimal or equilibrium solutions of an economic model vary relative to a perturbation of the model's parameters. Such predictions are important as they contain much of the empirical content of the economic model being studied.

The purpose of this paper is to propose a new approach to equilibrium comparative statics in economic environments where complementarities play a critical role, including environments in which the existing methods for obtaining monotone comparative statics appear inadequate. Such situations can arise in games with strategic complementarities (GSC), but also in other economic settings such as dynamic general equilibrium economies. To better explain the nature of the paper's methodological contribution, we start with a motivating example of a game that highlights both the limitations of the existing methods, as well as the contributions of our new comparative statics approach.[1]

Example 1. *Our leading example is a game with a continuum of actions and a continuum of equilibria. The ideas are easier to explain in this setting. However, analogous arguments apply to coordination games with a finite number of actions and multiple equilibria.*

Consider the following joint venture game. Players 1 and 2 choose actions a_1 *and* a_2*, respectively, interpreted as their effort. The cost of taking action* a_i *for player* $i = 1, 2$ *is* ca_i*, for some* $c \in (0, 1)$*. The output of the team that consists of the two players is* $2 \min\{a_1, a_2\}$*. So the payoff of each player* i *is* $\min\{a_1, a_2\} - ca_i$*. This game has a continuum of equilibria: All pairs* (a_1, a_2) *such that* $a_1 = a_2$ *are equilibrium strategies.*

Suppose that players are initially playing actions a_1^0 *and* a_2^0*, and the productivity of player 1 increases, so the output is* $2 \min\{ta_1, a_2\}$ *for some* $t > 1$*, and the payoffs are* $\min\{ta_1, a_2\} - ca_i$ *for* $i = 1, 2$*. Intuitively, the output should in-*

[1] This is a slightly modified version of the team of managers game studied in Milgrom and Roberts (1990), Section 4, Example (5).

crease. We cannot make this conclusion by comparing equilibria. For example, if $a_1^0 = a_2^0 = a^0 > 0$ *in the original equilibrium, the total output in this equilibrium (for* $t = 1$*) is equal to* a^0*. And for* $t > 1$ ***the game has a continuum of equilibria****.* ***In some of them the output is higher*** *than 1, but* ***in others the output is lower*** *than 1. In addition,* ***all equilibria are unstable.***

Suppose that players adaptively learn of playing this game with the new parameter t*. In discrete time, this learning happens through a sequence of actions* $(a^k)_{k=0}^{\infty}$*, starting from an action profile (not necessarily an equilibrium)* $a^0 = (a_1^0, a_2^0)$*. We consider the set* $\mathcal{S}(a^0)$ *of all "reasonable" sequences* $(a^k)_{k=0}^{\infty}$*, and define* $\underline{a}(t)$ *and* $\overline{a}(t)$ *such that in the long-run (i.e., for large* k*)* a^k *cannot be "much smaller" than* $\underline{a}(t)$ *and cannot be "much larger" than* $\overline{a}(t)$ *for any sequence* $(a^k)_{k=0}^{\infty} \in \mathcal{S}(a^0)$*.*

We compute that $\underline{a}(t)$ *and* $\overline{a}(t)$ *for this game are*

$$\begin{aligned} \underline{a}(t) &= (\min\{a_1^0, a_2^0/t\}, \min\{a_1^0 t, a_2^0\}), \\ \overline{a}(t) &= (\max\{a_1^0, a_2^0/t\}, \max\{a_1^0 t, a_2^0\}). \end{aligned} \tag{1}$$

Note that $\underline{a}(t)$ *and* $\overline{a}(t)$ *are equilibria. In general, this will not always be the case. One can check that in this example,*

$$br(\liminf_k a^k) = \liminf_k br(a^k),$$

where br *stands for the best response, for any sequence of action profiles* $(a^k)_{k=0}^{\infty}$*. We will prove later that if this (and an analogous condition for* $\limsup$*) are satisfied, then both our long-run bounds* $\underline{a}(t)$ *and* $\overline{a}(t)$ *are equilibria.*

So, suppose first that we started from an equilibrium $a_1^0 = a_2^0 = a^0$ *for* $t = 1$*, and we believe that the learning or adaptive dynamics after the change of* t *to* $t > 1$ *will end up in an equilibrium. The output in an equilibrium that can be reached after an increase in* t *to* $t > 1$ *is between* $2a^0$ *and* $2ta^0$*. That is,* ***the new equilibrium output*** *for* $t > 1$ ***is never lower than the output*** $2a^0$ ***in the original equilibrium****, but* ***it happens to be strictly higher for some new equilibria****.*

Moreover, we reached the conclusion regarding the total output, despite the fact that the action of at least one player cannot increase in the response to the *increase in t, and typically one of the two players takes a strictly lower action. Yet the total output is never lower.* ***Applying our approach, we can conclude, in contrast to the previous approaches, that the output will (weakly) increase in response to the increase in t.***

So far, we conducted equilibrium comparative statics. However, given the unstability of equilibria, it is plausible to expect chaotic dynamics that may not converge to any equilibrium. Our approach also applies to this case, and the conclusions seem more intriguing.

So, suppose that we start from an arbitrary action profile $a^0 = (a_1^0, a_2^0)$. Formulas (1) still apply, and we can derive from the the following claims:

(a) If $a_1^0 \leq a_2^0 \leq ta_1^0$, then the total output in $\underline{a}(t > 1)$ is $2a_2^0$, which is no less then $2a_1^0$ (the total output in $\underline{a}(t = 1)$).

(b) If $a_2^0 > ta_1^0$, then the total output in $\underline{a}(t > 1)$ is $2ta_1^0$, which is greater than $2a_1^0$ (the total output in $\underline{a}(t = 1)$).

(c) If $a_2^0 < a_1^0$, then the total output in $\underline{a}(t > 1)$ is $2a_2^0$, which is equal to the total output in $\underline{a}(t = 1)$).

Therefore, in each case we can conclude that the output will weakly increase in response to the increase in t.

Our approach to comparative statics can shortly be described as follows: Starting at an initial equilibrium (or actually, at an initially observed outcome), we identify sharp or tight bounds for remote iterations of a large class of adaptive learning sequences initiated by a perturbation of the model's parameters. We next compare the statics of interest at these bounds and at the initial equilibrium (or at the corresponding bounds for the original parameters). In our motivating example, these bounds for the initially observed outcome a^0 are $\underline{a}(t)$ and $\overline{a}(t)$ given in (1) (and the corresponding bounds for the original parameter are $\underline{a}(1)$ and $\overline{a}(1)$ when a^0 is not an equilibrium). If a statistic at $\underline{a}(t)$ is greater than that at a^0, then we claim that the statistic increases in response to the parameter change. For example, the total output increased in response to an increase from $t = 1$ to $t > 1$. If a^0 is not an equilibrium, then a conservative analyst may require the statistic at $\overline{a}(t)$ to be greater than that at the corresponding *lower* bound at a lower t to claim that the statistic increases in response to the parameter change. But a less conservative analysts may require only that the statistic at $\overline{a}(t)$ is greater than that at the corresponding *upper* bound and that the statistic at $\underline{a}(t)$ is greater than that at the corresponding *lower* bound for a lower parameter t.

Our approach is iterative and, methodologically, in the spirit of the celebrated "correspondence principle," a concept first presented in the work of Samuelson (1947), and then extended most notably in a series of papers by Echenique (e.g., Echenique (2002, 2004); see also McLennan (2015)). However, as the motivating example shows, it applies to environments with continuum of equilibria, which all can be unstable; and, as it will be shown later, it also applies to chaotic environments, in which the initially observed outcome may not be an equilibrium and adaptive learning sequences never converge.

Our propositions extend several existing results based upon fixed-point comparative statics for parameterized monotone correspondences in sigma-complete lattices, where the comparative statics results pertain to sets of fixed points, or only to extremal fixed points,[2] or to stable fixed points of Euclidean lattices (as in previously mentioned papers). In particular, we provide conditions to state comparative statics results for so called "mixed shocks," i.e., shocks affecting some variables positively and other variables negatively.

Related Literature For economic models in which the analysis reduces to solving to an optimization problem, there is a large set of comparative statics tools. They involve, among others, the implicit function theorem. These tools typically require strong regularity conditions on the optimization problem (e.g., the smoothness of objectives and constraints, the interiority of all optimal solutions, etc.), and comparative statics predictions are often only local in nature.[3] Alterna-

[2] Fixed-point comparative statics results for strong set order monotone (or "ascending") correspondences in complete lattices can be found in Topkis (1998), Chapter 2, Section 5. See also Theorem 2 in Chapter 10 in Veinott (1992). For comparative statics results on extremal fixed points, see Milgrom and Roberts (1990, 1994) and Milgrom and Shannon (1994).

[3] There are methods for globalizing the implicit function theorem. See the celebrated work of Gale and Nikaido (1965), as well as more recent contributions (and references therewithin) of Blot (1991), Phillips (2012), and Cristea (2017).

tively, lattice programming provides a set of tools for obtaining global monotone comparative statics of optimal solutions to parameter change (see Topkis (1998) among others).

Performing comparative statics analysis on equilibrium problems is more complicated. Especially in economic models with multiple equilibria, fixed-point comparative statics typically involves the tools of transversality and degree theory, and others from differential topology. These tools typically provide only weak local equilibrium comparative statics results, and even for these, they require stronger regularity conditions on the primitives than in the context of implicit function based comparative statics of optimization problems.[4] Alternatively, there is an extensive literature on fixed-point comparative statics for parameterized monotone operators and correspondences that transform chain-complete partially ordered sets. It is especially interesting about these tools that the equilibrium comparative statics are often computable. But a general limitation of these existing order-theoretic approaches is that they typically provide limited comparative statics information in the presence of multiple equilibria. That is, the comparative statics results pertain typically to only extremal equilibria (i.e., least/minimal or greatest/maximal) and the constructive nature of the comparative statics result does not hold for iterations from any initial point.[5]

A well-known approach to studying the equilibrium comparative statics of any equilibria is embodied in the so-called "Correspondence Principle," which was suggested originally in the seminal work of Samuelson (1947).[6] Here, one seeks to identify regularity conditions of optimization problems or equilibrium problems for unambiguous equilibrium comparative statics by refining away unstable equilibria, and then restricting attention to regular (or smooth) equilibria. This approach

[4] There is an extensive literature on these approaches to equilibria comparative statics for "regular economies" based upon versions of Thom's transversality theory and Sard's theorem.

For surveys of work on regular economies, see Mas-Colell (1985, 1996), Nagata (2004), and McLennan (2018).

[5] See, for example, the computable comparative statics results for Nash equilibrium in Bayesian supermodular games in Van Zandt (2010). Also, see the discussion in Balbus et al. (2022a). There, iterations need to start from least (resp., greatest) elements of the domain.

[6] See also McLennan (2015) for an interesting recent discussion of the correspondence principle, citations of the extensive literature and implications for equilibrium comparative statics.

is inherently dynamic, and can be applied when equilibria are locally unique and amenable to applications of the implicit function theorem. Echenique (2002) has extended these ideas substantially, and has been able to prove stronger versions of the Correspondence Principle for GSC's on lattices A when there is a convex set of parameters T. For example, Echenique (2002) showed that in GSC's, a continuous equilibrium selector $t \to a^*(t)$ is increasing if and only if it selects stable equilibria.[7]

Our paper shares with the correspondence principle the idea that the identification of monotone comparative statics is critically tied to a dynamic approach. That is, one is interested in viewing an equilibrium as the stationary point of a dynamical system, in which a new equilibrium emerges from an old equilibrium after a change in a parameter value via some dynamic adjustment process. For example, if an equilibrium at the original set of parameters is locally stable, then one can develop sufficient conditions on the behavior of this dynamical system that guarantee that starting from the equilibrium for the old parameter, the dynamical system will actually converge to the new equilibrium for small changes of the parameter.

But this leaves open many interesting questions. Aside from the obvious question of relaxing the needed topological structure and conditions required to study the stability of local equilibrium comparative statics via correspondence principle based arguments, what do we do when all the equilibria are unstable? What if there is a continuum of equilibria (i.e., equilibria are indeterminate and not locally isolated)? Or finally, what if the setting is chaotic, and observed outcomes may not converge to any equilibrium? Echenique (2002) provides a partial answer to these questions by observing that in GSC's, if a correspondence $BR : A \to A$ defined on the space of action profiles A, a complete lattice, is strongly increasing and upper hemi-continuous, then for every action profile a^0 such that $a^0 \leq \inf BR(a^0)$, a best-response sequence starting from a^0 obviously converges to a fixed point of

[7] See also Echenique (2002, 2004) for the precise formulations of various versions of this result. Notice, in the presence of multiple equilibria, the existence of continuous equilibrium selector is an added complication in applying Echenique (2002) results. But he is able to weaken the continuity requirements in some cases.

BR that is higher than a^0.[8] In this paper we provide answers to these questions, in particular, by substantially generalizing Echenique's result.

The remainder of the paper is organized as follows. In the next section, we define mathematical terminology. In section 3, we introduce our main concepts. Sections 4 and 5 contain examples and main results. The results that explain the relation of our approach to the existing results are in Section 4, and newer results on mixed shocks and aggregate comparative statics are in Section 5. In Section 6, we present applications. We include the proofs into the main text, delegating only the proofs of more technical results to Appendix.

2 Preliminaries

We start with introducing some basic definitions. A partially ordered set (or *poset*) is set A equipped with a partially order $\geq$. For $a', a \in A$, we say a' is *strictly higher* than a, and write $a' > a$, whenever $a' \geq a$ and $a' \neq a$. A poset $(A, \geq)$ is a *lattice* if for any $a, a' \in A$ there exists the *join* $a \vee a' := \sup\{a, a'\} \in A$ and there exists the *meet* $a \wedge a' := \inf\{a, a'\} \in A$. A lattice A is *complete (resp. sigma-complete)* if there exists $\bigvee B := \sup B \in A$ and $\bigwedge B := \inf B \in A$, for any (resp. countable) $B \subseteq A$. A subset $B \subset A$ is a *sublattice* of A if B is a lattice in the order induced from A, i.e. the join $a \vee a'$ and the meet $a \wedge a'$ as defined in $(A, \geq)$ belongs to B for all $a, a' \in B$.

Let $(A, \geq)$ and $(B, \geq)$ be posets. A mapping $f : A \to B$ is order-preserving (or increasing) on A if $a' \geq a$ implies $f(a') \geq f(a)$ for a, a' in A. Let $F : A \rightrightarrows B$ be a nonempty-valued correspondence. Assume that for each $a \in A$ set $F(a)$ has the greatest and the least element and denote them with $\overline{F}(a) := \sup F(a)$ and

[8] See Milgrom and Roberts (1990) and Milgrom and Shannon (1994) for some results preceding Echenique's result. See also Balbus et al. (2022a) Proposition A.2. for some recent generalizations of monotone comparative statics results for dynamic games. Finally, see Heikkilä and Reffett (2006) for fixed-point results for parameterized correspondences with applications to games (and in particular, GSC's).

$\underline{F}(a) := \inf F(a)$. We say F is *weakly increasing*[9] whenever $a' \geq a$ implies that

$$\overline{F}(a') \geq \overline{F}(a) \text{ and } \underline{F}(a') \geq \underline{F}(a).$$

We say F is *strongly increasing* whenever $a' \geq a$ implies that

$$\underline{F}(a') \geq \overline{F}(a).$$

A sequence $(a^k)_{k=0}^{\infty}$ of elements of A is *monotone increasing* if $a^{k+1} \geq a^k$ for each k. It is *strictly increasing* if $a^{k+1} > a^k$ for each k. *Monotone decreasing* and *strictly decreasing* can be defined in the obvious dual manner. A *monotone sequence* then is either monotone increasing or monotone decreasing. We say that a monotone increasing (resp., monotone decreasing) sequence $(a^k)_{k=0}^{\infty}$ *converges* to $a \in A$ whenever $\bigvee_{k \geq 0} a^k = a$ (resp., $\bigwedge_{k \geq 0} a^k = a$).[10] That is, when a is the supremum (resp., infimum) of the monotone increasing (resp., monotone decreasing) sequence.

Suppose A and B are sigma-complete lattices. The mapping $f : A \to B$ is *upward order continuous* (resp., *downward order continuous*) if for any increasing sequence (a^k) with $a^k \in A$, we have:

$$f\left(\bigvee_{k \geq 0} a^k\right) = \bigvee_{k \geq 0} f(a^k) \quad \left(\text{respectively } f\left(\bigwedge_{k \geq 0} a^k\right) = \bigwedge_{k \geq 0} f(a^k)\right).$$

The mapping f is then *order continuous* if it is both upward and downward order continuous. Notice, if f is upward (resp., downward) order continuous, it is order-preserving or monotone increasing function on A.[11]

[9] Observe that weakly monotonicity of a correspondence is a weaker notion than (Veinott-) strong set-order monotonicity. In particular, $F(a)$ need not be sublattice valued.

[10] Notice that in the definition of convergence of monotone sequences convergence is in *order*.

[11] If a mapping is upward (resp., downward) order continuous, it is also by definition sup (resp, inf) preserving. So our definitions here coincide with standard definitions of order continuity (e.g., Dugundji and Granas (1982), p 15).

3 Concepts

3.1 Adaptive learning

The idea of our approach to comparative statics can be described as follows. Suppose that we initially observe variables or actions to be a^0. This a^0 need not even be a Walrasian or Nash equilibrium, or an equilibrium in any other sense if we expect chaotic behavior in the studied setting. Then a parameter of the setting changes, and this initiates adaptive learning in the new setting. If we allow for chaotic behavior, then the adaptive learning applies also to the original setting, before the parameter changed. It seems reasonable to assume that this learning must have the form of some sequence $(a^k)_{k=0}^{\infty}$ from a class of sequences $\mathcal{S}(a^0)$. We will focus on the following, large class of sequences $\mathcal{S}(a^0)$. However, we emphasize that analysts are allowed to choose their preferred class of sequences.

For a correspondence $F : A \rightrightarrows A$, a sequence $(a^k)_{k=0}^{\infty}$ starting from a^0 is an adaptive learning sequence if

$$\exists_{\gamma \in \mathbb{N}} \, \forall_{k \in \mathbb{N}} \; \underline{F}(\inf\{a^k, ..., a^{k-\gamma+1}\}) \leq a^{k+1} \leq \overline{F}(\sup\{a^k, ..., a^{k-\gamma+1}\}),$$

where it is assumed $a^{k-\gamma+1} = a^0$ for $k < \gamma - 1$. Denote by $\mathcal{S}(a^0)$ the set of all *adaptive learning sequences* starting from a^0.

The idea is to be agnostic about the specific form of learning. We basically impose one postulate, namely, that players respond optimally to some history of the actions observed in previous, but not too remote periods. The parameter γ measures how far players look into the past, and $\inf\{a^k, ..., a^{k-\gamma+1}\}$ and $\sup\{a^k, ..., a^{k-\gamma+1}\}$ are the extreme statistics of the actions they observed in the past they look into.

Echenique (2002) used *convergent* sequences $(a^k)_{k=0}^{\infty} \in \mathcal{S}(a^0)$ for conducting comparative statics within equilibrium setting. However, as we demonstrate the comparative statics can be conducted even when there is no convergent adaptive learning sequence.

3.2 Bounds of learning sequences

Suppose A is a sigma-complete lattice. At first sight, studying all sequences $(a^k)_{k=0}^{\infty} \in \mathcal{S}(a^0)$ seems intractable. However, we will show next that it reduces to studying single action profiles $\underline{a}$ and $\overline{a}$. To define these action profiles, we first define by induction, for any given setting, action profiles $\underline{a}^{k,\gamma}$ (and simultaneously, action profiles $\overline{a}^{k,\gamma}$): Let $\underline{a}^{0,\gamma} = \overline{a}^{0,\gamma} = a^0$ for all $\gamma \in \mathbb{N}$, and let

$$\underline{a}^{k+1,\gamma} = \underline{F}(\inf\{\underline{a}^{k,\gamma}, ..., \underline{a}^{k-\gamma+1,\gamma}\})$$

and

$$\overline{a}^{k+1,\gamma} = \overline{F}(\sup\{\overline{a}^{k,\gamma}, ..., \overline{a}^{k-\gamma+1,\gamma}\}).$$

It is assumed that $\underline{a}^{k-l,\gamma} = \underline{a}^{0,\gamma}$ and $\overline{a}^{k-l,\gamma} = \overline{a}^{0,\gamma}$ for $l > k$. Next, for any given γ, define

$$\liminf_k \underline{a}^{k,\gamma} = \bigvee_k \bigwedge_{l \geq k} \underline{a}^{l,\gamma}$$

and

$$\limsup_k \overline{a}^{k,\gamma} = \bigwedge_k \bigvee_{l \geq k} \overline{a}^{l,\gamma}$$

Observe that $\liminf_k \underline{a}^{k,\gamma}$ and $\limsup_k \overline{a}^{k,\gamma}$ exist by sigma-completeness of the lattice. In addition, sequence $\left(\liminf_k \underline{a}^{k,\gamma}\right)_{\gamma=0}^{\infty}$ is decreasing and the sequence $\left(\limsup_k \overline{a}^{k,\gamma}\right)_{\gamma=0}^{\infty}$ is increasing. Let $\underline{a} = \lim_{\gamma\to\infty} \liminf_k \underline{a}^{k,\gamma}$ and $\overline{a} = \lim_{\gamma\to\infty} \liminf_k \overline{a}^{k,\gamma}$.

The following table contains a useful graphical exhibition of sequences $\underline{a}^{k,\gamma}$, limits $\liminf_k \underline{a}^{k,\gamma}$ and profile $\underline{a}$, as well as the inequalities and convergences between them. An analogous graphical exhibition applies to $\overline{a}^{k,\gamma}$, $\liminf_k \overline{a}^{k,\gamma}$ and $\overline{a}$.

$$\begin{array}{ccccccccc}
\liminf_k \underline{a}^{k,1} & & \underline{a}^{4,1} & & \underline{a}^{3,1} & & \underline{a}^{2,1} & & \underline{a}^{1,1} \\
\geq & & \geq & & \geq & & \geq & & \geq \\
\liminf_k \underline{a}^{k,2} & & \underline{a}^{4,2} & & \underline{a}^{3,2} & & \underline{a}^{2,2} & \leq & \underline{a}^{1,2} \\
\geq & & \geq & & \geq & & \geq & & \geq \\
\liminf_k \underline{a}^{k,3} & & \underline{a}^{4,3} & & \underline{a}^{3,3} & \leq & \underline{a}^{2,3} & \leq & \underline{a}^{1,3} \\
\geq & & \geq & & \geq & & \geq & & \geq \\
\liminf_k \underline{a}^{k,4} & & \underline{a}^{4,4} & \leq & \underline{a}^{3,4} & \leq & \underline{a}^{2,4} & \leq & \underline{a}^{1,4} \\
\downarrow \gamma & & & & & & & & \\
\underline{a} & & & & & & & &
\end{array}$$

3.3 General comparative statics

In applications, we are typically interested in comparing some statistic $\varphi(a) \in \mathbb{R}$ in the long-run for the original and new setting, that is, in comparing $\varphi(a^k)$ for large enough values of k. In the realm of monotone mappings f or correspondences F, it makes sense to limit attention to monotone statistics φ. In our motivating example, φ was the total output. Intuitively, if $\varphi(a^{new,k})$ for any large value of k in the new setting is at worst only slightly smaller than $\varphi(a^{old,k})$ for any large value of k in the original setting for any sequences $(a^{old,k})_{k=0}^{\infty}, (a^{new,k})_{k=0}^{\infty} \in \mathcal{S}(a^0)$ provided that k is large enough, and this "slightly" is closer and closer to zero for larger k, then we can say that in the long-run the statistic φ increases in response to the parameter change.

We are now ready to define monotone comparative statics in the most general case. To formalize the dependence on a parameter, we introduce a set of parameters T, and assume that correspondence $F : A \times T \rightrightarrows A$. By $\underline{a}^{k+1,\gamma}(t)$ and $\overline{a}^{k+1,\gamma}(t)$ we denote elements of the sequences constructed in Section 3.2 for $F(\cdot, t)$. We analogously define $\underline{a}(t)$ and $\overline{a}(t)$. Let function $\varphi : A \times T \to \mathbb{R}$ denote an aggregate. We now study only correspondences F and functions φ that are monotone in a for any given t.

Definition 1. *A statistic $\varphi : A \times T \to \mathbb{R}$ weakly increases with a parameter change from t' to t'' if $\varphi(\underline{a}(t''), t'') \geq \varphi(\underline{a}(t'), t')$ as well as $\varphi(\overline{a}(t''), t'') \geq \varphi(\overline{a}(t'), t')$.*

We say such a statistic increases strongly with a parameter change if $\varphi(\underline{a}(t''), t'') \geq \varphi(\overline{a}(t'), t')$.

We can define in a dual manner that a statistic decreases. We will now motivate Definition 1. Suppose that we initially observe a^0 given a parameter $t = t'$ or t''. Let for a moment $\underline{a} := \underline{a}(t)$ and $\overline{a} := \overline{a}(t)$ and similarly $\underline{a}^{k,\gamma} := \underline{a}^{k,\gamma}(t)$ and $\overline{a}^{k,\gamma} := \overline{a}^{k,\gamma}(t)$. Since $\underline{a} \leq \liminf_k \underline{a}^{k,\gamma}$ for any γ, $\liminf_k \underline{a}^{k,\gamma}$ is the limit of the increasing sequence $\left(\inf_{l \geq k} \underline{a}^{l,\gamma}\right)_{l=0}^{\infty}$. Therefore, because we consider continuous comparative statics, for any $\varepsilon > 0$ and for any large enough value of k, it must be that $\varphi(\underline{a}) - \varepsilon < \varphi\left(\inf_{l \geq k} \underline{a}^{l,\gamma}\right)$, which in turn implies that $\varphi(\underline{a}) - \varepsilon < \varphi(\underline{a}^{l,\gamma})$ for

$l \geq k$. Similarly, for any $\varepsilon > 0$ and for any large enough value of k, it must be that $\varphi(\overline{a}^{l,\gamma}) < \varphi(\overline{a}) + \varepsilon$ for $l \geq k$.

Therefore, if $\varphi(\underline{a}(t''), t'') \geq \varphi(\overline{a}(t'), t')$, then $\varphi(\underline{a}^{l,\gamma}(t'), t')$ can be larger only a little than $\varphi(\overline{a}^{l,\gamma}(t''), t'')$ for large l. So, "at infinity," $\varphi(a^k(t'), t') \leq \varphi(a^k(t''), t'')$ for any adaptive learning sequences $(a^k(t'))_{k=0}^{\infty}$ and $(a^k(t''))_{k=0}^{\infty}$. Similarly, if $\varphi(\overline{a}(t''), t'') \leq \varphi(\underline{a}(t'), t')$, then $\varphi(a^k(t'), t')) \geq \varphi(a^k(t''), t'')$ at infinity for any adaptive learning sequences $(a^k(t'))_{k=0}^{\infty}$ and $(a^k(t''))_{k=0}^{\infty}$.

Note that our approach to comparative statics is conservative. Indeed, $\varphi(a^0) \leq \varphi(\underline{a})$ implies that $\varphi(a^0) - \varepsilon < \varphi(a^{k,\gamma})$ for large enough values of k. However, the converse is in general false. This converse implication would be true if we were comparing the elements of the lattice A instead of thereof statistic. More precisely, suppose that $b < \underline{a}^{k,\gamma}$ for any $b < a^0$ and for large enough values of k. Then $b \leq \inf_{k \geq K} \underline{a}^{k,\gamma}$ for large values of K. So $b \leq \liminf_k \underline{a}^{k,\gamma}$ for all γ, and this implies that $b \leq \underline{a}$.

3.4 Equilibrium analysis

So far our comparative statics was not an equilibrium analysis. One may wish to postulate, however, that adaptive learning should result in the long -run in an equilibrium. Or one may simply be interested only in equilibrium analysis.[12] Under some seemingly natural conditions, the bounds $\underline{a}$ and $\overline{a}$ are in fact fixed points of F. Thus, if these conditions are satisfied, our bounds are appropriate tools for equilibrium comparative statics. Suppose $f : A \to A$ is a monotone increasing mapping.

Theorem 1. *If $f(\liminf_k a^k) = \liminf_k f(a^k)$ for any sequence $(a^k)_{k=0}^{\infty}$, then $\underline{a}$ is a fixed point. If $f(\limsup_k a^k) = \limsup_k f(a^k)$ for any sequence $(a^k)_{k=0}^{\infty}$, then $\overline{a}$ is a fixed point.*

We will prove the first part of the theorem. The proof of the second part is analogous.

[12] We refer the reader to our related paper Balbus et al. (2022b) and Olszewski (2021a,b) for results on tight fixed point (equilibrium) bounds of best response iterations on monotone correspondences and functions.

Claim 1. *$\underline{a}^{k+1,\gamma+1} \leq f(\underline{a}^{k,\gamma})$ for all k and γ.*

Proof. Since $\underline{a}^{k,\gamma+1} \leq \underline{a}^{k,\gamma}$ for all k and γ,

$$\underline{a}^{k+1,\gamma+1} = f(\min\{\underline{a}^{k,\gamma+1}, ..., \underline{a}^{k-\gamma,\gamma+1}\}) \leq f(\min\{\underline{a}^{k,\gamma+1}, ..., \underline{a}^{k-\gamma+1,\gamma+1}\})$$
$$\leq f(\min\{\underline{a}^{k,\gamma}, ..., \underline{a}^{k-\gamma+1,\gamma}\}) \leq f(\underline{a}^{k,\gamma}).$$

□

Observe now that

$$f(\underline{a}) = \lim_{\gamma} f\left(\liminf_{k} \underline{a}^{k,\gamma}\right) = \lim_{\gamma}\left[\liminf_{k} f\left(\underline{a}^{k,\gamma}\right)\right]$$
$$\geq \lim_{\gamma}\left[\liminf_{k} \underline{a}^{k+1,\gamma+1}\right] = \lim_{\gamma}\left[\liminf_{k} \underline{a}^{k,\gamma}\right] = \underline{a}.$$

The first two equalities follow from the definition of $\underline{a}$ and our assumption; in particular, $\left(\liminf_k \underline{a}^{k,\gamma}\right)_{\gamma=1}^{\infty}$ is a decreasing sequence, whose limit is $\underline{a}$. The inequality follows from Claim 1, and the last two equalities follow directly from definitions.

Claim 2. *$f(\underline{a}^{k,\gamma+1}) \leq \underline{a}^{k,\gamma}$ for all k, γ*

Proof. Since $\underline{a}^{k,\gamma+1} \leq \underline{a}^{k,\gamma}$ for all k and γ,

$$f(\underline{a}^{k,\gamma+1}) = f(\min\{\underline{a}^{k-1,\gamma+1}, ..., \underline{a}^{k-\gamma-1,\gamma+1}\}) \leq f(\min\{\underline{a}^{k-1,\gamma+1}, ..., \underline{a}^{k-\gamma,\gamma+1}\})$$
$$\leq f(\min\{\underline{a}^{k-1,\gamma}, ..., \underline{a}^{k-\gamma,\gamma}\}) = \underline{a}^{k,\gamma}.$$

□

Observe finally that

$$f(\underline{a}) = \lim_{\gamma} f\left(\liminf_{k} \underline{a}^{k,\gamma+1}\right) = \lim_{\gamma}\left[\liminf_{k} f\left(\underline{a}^{k,\gamma+1}\right)\right]$$
$$\leq \lim_{\gamma}\left[\liminf_{k} \underline{a}^{k,\gamma}\right] = \underline{a}.$$

The equalities follow from the same argument as that in the previous display, and the inequality follows from Claim 2.

Theorem 1 generalizes to weakly increasing correspondences $F : A \rightrightarrows A$ such that $\overline{F}(\limsup_k a^k) = \limsup_k \overline{F}(a^k)$ and $\underline{F}(\liminf_k a^k) = \liminf_k \underline{F}(a^k)$.

Therefore, we suggest the following concept for the settings that satisfy the continuity conditions from Theorem 1.

Definition 2. *Suppose a^0 is an equilibrium given some $t \in T$. A statistic $\varphi : A \times T \to \mathbb{R}$ increases in equilibrium with a parameter change from t' to t'' if $\varphi(\underline{a}, t'') \geq \varphi(a^0, t')$, and it decreases with a parameter change if $\varphi(\overline{a}, t'') \leq \varphi(a^0, t')$, where $\underline{a} = \underline{a}(t'')$ and $\overline{a} = \overline{a}(t'')$ and $\underline{a}(t') = \overline{a}(t') = a^0$.*

We conclude this section with an example that shows that $\underline{a}$ and $\overline{a}$ need not be fixed points in the general case, that is, for all monotone increasing mappings f.

Example 2. *Let $A = \{(-1/n, 1/m) : n, m = 1, 2, ...$ and $n \leq m\} \cup \{(-1/n, 0) : n = 1, 2, ...\} \cup \{(0, 1)\} \cup \{(0, 0)\}$ be the lattice equipped with the ordering inherited from $\mathbb{R}^2$. Let $f : A \to A$ be the mapping defined by letting*

$$f(-1/n, 1/m) = (-1/(n+1), 1/(m+1)),$$

$$f(-1/n, 0) = f(0, 0) = (-1, 0).$$

Then, A is a complete lattice, and f is an order-preserving, continuous mapping. Suppose that $a^0 = (-1, 1)$ then $\underline{a} = (0, 0)$ but $f(\underline{a}) = (-1, 0)$.

4 Analysis: examples and results

We begin this section with returning to our motivating example.

Example 1 (continued). We will now show the computations that we referred to but skipped in the introduction. For $t = 1$, every sequence $(a^k)_{k=0}^{\infty} \in \mathcal{S}(a^0)$, where $a^0 = (a_1^0, a_2^0)$, is contained in the square $[\min\{a_1^0, a_2^0\}, \max\{a_1^0, a_2^0\}]^2$. Indeed, this is clearly true for $a^{0,\gamma}$ for any γ. Suppose this is true for $a^{l,\gamma}$, where $l = 1, ..., k$. This implies that

$$\min\{a_1^0, a_2^0\} \leq \min\{a_i^{k,\gamma}, ..., a_i^{k-\gamma,\gamma}\} \leq \max\{a_i^{k,\gamma}, ..., a_i^{k-\gamma,\gamma}\} \leq \max\{a_1^0, a_2^0\},$$

and this in turn implies that

$$\min\{a_1^0, a_2^0\} \leq br(\min\{a_i^{k,\gamma}, ..., a_i^{k-\gamma,\gamma}\}) \leq a_j^{k+1,\gamma} \leq br(\max\{a_i^{k,\gamma}, ..., a_i^{k-\gamma,\gamma}\}) \leq \max\{a_1^0, a_2^0\}$$

for $i = 1, 2$ and $j = 3 - i$.

In addition, there is a sequence $(a^{k,\gamma})_{k=0}^{\infty} \in \mathcal{S}(a^0)$ such that $a^{k,\gamma} = (\min\{a_1^0, a_2^0\}, \min\{a_1^0, a_2^0\})$ for $k = 2, 3, ...$ and $\gamma \geq 2$, and there is a sequence $(a^{k,\gamma})_{k=0}^{\infty} \in \mathcal{S}(a^0)$ such that $a^{k,\gamma} = (\max\{a_1^0, a_2^0\}, \max\{a_1^0, a_2^0\})$ for $k = 2, 3, ...$ and $\gamma \geq 2$. We will construct such a sequence for min; the argument for max is analogous. Let:

$$a_j^{1,\gamma} = br(a_i^0) = a_i^0,$$

$$a_j^{2,\gamma} = br(\min\{a_i^0, a_i^{1,\gamma}\}) = br(\min\{a_1^0, a_2^0\}) = \min\{a_1^0, a_2^0\},$$

and more generally,

$$a_j^{k+1,\gamma} = br(\min\{a_i^{k,\gamma}, ..., a_i^{k-\gamma,\gamma}\}) = br(\min\{a_1^0, a_2^0\}) = \min\{a_1^0, a_2^0\}$$

for $k = 1, 2, ...$

Therefore

$$\underline{a}(1) = (\min\{a_1^0, a_2^0\}, \min\{a_1^0, a_2^0\}) \text{ and } \overline{a}(1) = (\max\{a_1^0, a_2^0\}, \max\{a_1^0, a_2^0\}).$$

Consider now the comparative statics exercise in which the productivity of player 1 increases, so the output is $2\min\{ta_1, a_2\}$ for some $t > 1$. By an argument analogous to that used to compute $\underline{a}(1)$, we compute that for the new productivity

$$\begin{aligned} \underline{a}(t) &= (\min\{a_1^0, a_2^0/t\}, \min\{a_1^0 t, a_2^0\}) \\ \overline{a}(t) &= (\max\{a_1^0, a_2^0/t\}, \max\{a_1^0 t, a_2^0\}). \end{aligned}$$

This enables us to derive the results that we announced in the introduction. For example, the total output is $2a_1^0$ in the original equilibrium (i.e. $a_1^0 = a_2^0$), and this is equal to the total output at $\underline{a}(t)$.

4.1 Samuelson's Correspondence Principle

Consider now a classic example of GSC, namely, the Bertrand competition with heterogenous products. This is an example of a setting to which Samuelson (1947) Correspondence Principle applies. We will discuss this principle after presenting the example, and show that our approach to comparative statics generalizes Samuelson's principle.

Example 3. *Two firms compete in prices. The reaction curve of firm* $i = 1, 2$ *to the price of firm* $j \neq i$ *is given by*

$$p_i = a + bp_j,$$

where $1/b > b$.[13] *This last inequality means that the reaction curve of firm 1 is steeper than the reaction curve of firm 2 in the system of coordinates with* p_1 *on the horizontal axis and* p_2 *on the vertical axis.*

For $\gamma = 1$, *our class of dynamics reduces to the best-response dynamic. So, it is well-known that the best-response sequence* $(p^{k,1})_{k=0}^{\infty}$ *converges to* (p_1^*, p_2^*), *where*

$$p_1^* = p_2^* = \frac{a}{1-b}$$

are the unique Nash equilibrium prices. Assume that $\gamma = 2$, *and that* p^0 *lies between the two reaction curves in the system of coordinates* p_1 *and* p_2. *We will show that any sequence* $(p^{k,2})_{k=0}^{\infty}$ *from our class of sequences* $\mathcal{S}(p^0)$ *also converges to* (p_1^*, p_2^*). *The argument for an arbitrary* $\gamma \geq 2$ *is analogous.*

If p^0 *lies between the two reaction curves, then the sequence* $(p^k)_{k=0}^{\infty}$ *generated by the best-response dynamic is monotonic. The assumption on* p^0 *is also inessential, but the argument is slightly more involved in other cases. Say, it is decreasing, that is,* $p_i^0 > p_i^*$ $i = 1, 2$. *We show by induction that*

$$\overline{p}^{k,2} = p^{k-1} \text{ and } \underline{p}^{k,2} = p^k \tag{2}$$

for $k \geq 2$.

Indeed, since $\overline{p}^{1,2} = \underline{p}^{1,2} = p^1$ *and* $p_i^1 \leq p_i^0$ *for* $i = 1, 2$, *we have that* $\underline{p}_i^{2,2} = br(p_j^1) = p_i^2$ *and* $\overline{p}_i^{2,2} = br(p_j^0) = p_i^1$. *So, (2) holds for* $k = 2$. *Similarly, since* $p_i^k \leq p_i^{k-1}$ *for* $i = 1, 2$, *we obtain, by (2) for* k, *that* $\underline{p}_i^{k+1,2} = br(p_j^k) = p_i^{k+1}$ *and* $\overline{p}_i^{k+1,2} = br(p_j^{k-1}) = p_i^k$.

Obviously, (2) implies that any sequence $(p^{k,2})_{k=0}^{\infty}$ *from* $\mathcal{S}(p^0)$ *converges to* (p_1^*, p_2^*).

Samuelson (1947) argues that for a stable equilibrium of a smooth mapping f, one obtains local comparative statics by referring to the Implicit Function Theorem. In the setting studied in our paper, stability would be naturally interpreted

[13] For smplicity, we restrict attention to two symmetric firms, but this is inessentioal.

as convergence to an equilibrium a^* of all adaptive learning sequences starting at an a^0 which is close to a^*. Up to now, we have studied only convergence of monotone sequences. However, as Example 3 shows, there typically exist non-monotone convergent sequences. Convergence of monotone sequences would not eliminate "saddle-path" equilibria. Therefore, we need to extend our definition of a convergent sequence. For comparison with Samuelson (1947) we restrict attention in this subsection to $A \subseteq \mathbb{R}^l$ with a usual coordinate-by-coordinate order. By $a' \gg a$ we mean that $a'_i > a_i$ for each $i = 1, 2, \ldots, l$.

Definition 3. *A sequence $(a^k)_{k=0}^{\infty}$ of elements of a lattice converges to an element a^* if*

$$\forall_{a \ll a^*} \ \exists_k \ \forall_{l \geq k} \ a^l > a$$

and

$$\forall_{a \gg a^*} \ \exists_k \ \forall_{l \geq k} \ a^l < a.$$

Theorem 2. *Suppose that*

$$\sup\{a \in A : a^* \gg a\} = a^* = \inf\{a \in A : a^* \ll a\}. \tag{3}$$

If all adaptive learning sequences $(a^k)_{k=0}^{\infty}$ converge to a^*, then $\underline{a} = \overline{a} = a^*$.

Theorem 2 implies an important corollary that our approach is consistent with Samuelson's Correspondence Principle. To see why, notice first that condition (3) is satisfied in the settings considered by Samuelson, because a smooth mapping f considered by him is defined on an open set in an Euclidean space, considered with the standard coordinate-by-coordinate order. Suppose that a^0 from an open set is an equilibrium for a parameter t', and consider a small change of the parameter to t''. Since the change is small, there should be a stable equilibrium a^* in the open set for parameter t'', and a^0 should belong to the basin of attraction of a^*. Thus, Theorem 2 implies that according to our approach the comparative statics reduces to comparing a^0 and $\underline{a} = \overline{a} = a^*$, exactly as suggested by Samuelson.

We relegate the proof of Theorem 2 to Appendix.

Remark 1. *Notice that the condition* $\underline{a} = \overline{a} = a^*$ *is actually close to being sufficient for, but not equivalent to, the stability of* a^**, defined as convergence to* a^* *of all adaptive learning sequences starting at an* a^0*. Indeed, if* $\underline{a} = a^*$ *and* $a < \underline{a}$*, then* $a < \liminf_k \underline{a}^{k,\gamma}$ *for every* γ*, because* $\underline{a} \leq \liminf_k \underline{a}^{k,\gamma}$*. Since* $\liminf_k \underline{a}^{k,\gamma} = \bigvee_k \bigwedge_{l \geq k} \underline{a}^{l,\gamma}$*, the last inequality often means that* $a < \bigwedge_{l \geq k} \underline{a}^{l,\gamma}$ *for any large enough value of* k*. And if this is true, then* $a < a^{l,\gamma}$ *whenever* $l \geq k$ *for any adaptive learning sequence* $(a^{k,\gamma})_{k=0}^{\infty}$*. However, in general* $a < \bigvee_k \bigwedge_{l \geq k} \underline{a}^{l,\gamma}$ *does not imply that* $a < \bigwedge_{l \geq k} \underline{a}^{l,\gamma}$ *for any large enough value of* k*.*[14]

4.2 Generalizing Echenique Results

Let $\underline{a}(t)$ and $\overline{a}(t)$ be the bounds of adaptive learning sequences constructed in Section 3.2 starting from a^0 and iterating on $F(\cdot, t)$ for given $t \in T$.

Theorem 3. *Let* A *be a sigma-complete lattice and* T *be a poset. Endow* $A \times T$ *with the product order. Let* $F : A \times T \rightrightarrows A$ *be such that* $F(a, t)$ *has the greatest and the least elements for each* a, t *and suppose* F *is weakly increasing on* $A \times T$*. If* $t' < t''$*, then:*

(i) $\underline{a}(t') \leq \underline{a}(t'')$ *and* $\overline{a}(t') \leq \overline{a}(t'')$*;*

(ii) if a^0 *is a fixed point of* $F(\cdot, t')$ *and for any* $a \in A$*,* $F(a, \cdot)$ *is strongly increasing, then additionally* $a^0 \leq \underline{a}(t'')$*.*

The central result of Theorem 3 is point (i). It assures that the bounds of iterations starting from any initial a^0 are ordered with respect to t. Weak monotonicity of F suffices to assure this weak monotone comparative statics result. Point (ii) provides a strong monotone comparative statics if iterations start from a fixed point a^0. Observe, the condition in (ii) on a correspondence F is always

[14] For example, a from $\mathbb{R}^2$ may have one coordinate smaller than, and the other coordinate equal to the join meet, but a may have the former coordinate smaller than the first coordinates of all meets, but the latter coordinate greater.

satisfied if it is a function.[15]

Focusing on GSC allows us to compare our results to those based on the correspondence principle in Echenique (2002). Under his assumption that $a^0 \leq \inf F(a^0, t)$, the smallest equilibrium which is the limit of a convergent sequences $(a^k)_{k=0}^{\infty}$ coincides with our lower bound $\underline{a}(t)$. But our result extends Echenique's result in few dimensions: (i) our correspondence F is assumed to be only weakly (not necessarily strongly) increasing; (ii) the adaptive dynamics may start from an action profile a^0 that is not ordered with its image[16] under F; (iii) the initial action profile a^0 need not be an equilibrium. Further, the adaptive dynamics may or may not be *convergent.*

Observe that Theorem 3 does not require any continuity of a correspondence F. This is in a stark difference with respect to monotone comparative statics results based upon the application of Tarski-Kantorovitch fixed point theorem on sigma-complete posets (see e.g. Balbus et al. (2022a) Proposition A.2.). So relaxing the demand for a theory of equilibrium comparative statics to comparative statics of iterative bounds, allows us to obtain a new comparative statics result for discontinuous correspondences. Moreover, observe that Theorem 3 applies to sets A that are only sigma-complete. This is an important generalization with respect to comparative statics results of Veinott (1992) (Theorem 2 in Chapter 10)[17] that require A to be a complete lattice, correspondence F to be strong set order monotone and pertain to comparative statics of extremal fixed points only. Again, Theorem 3 is particularly useful in environments in which comparative statics of iterative bounds is more adequate than comparative statics of extremal equilibria (see examples in Section 6).

Proof. Proof of (i). By the monotonicity of F, $\inf F(a^0, t') \leq \inf F(a^0, t'')$. So, $\underline{a}^{1,\gamma}$

[15] It is important to keep in mind that Theorem 3 itself does not guarantee existence of fixed points of $F(\cdot, t)$. If A is assumed to be additionally complete, then under conditions of Theorem 3 the set of fixed points of $F(\cdot, t)$ is nonempty for each t by Tarski's theorem. When A is only sigma-complete, sufficient conditions for the existence of fixed points of $F(\cdot, t)$ can also be provided. See Balbus et al. (2022b) and a discussion in Section 6.4.

[16] We find point (ii) important, because the Echenique assumption refers to a property of the initial equilibrium (or the starting point), not to a feature of the setting.

[17] See also Topkis (1998) Theorem 2.5.2.

for t' is no greater than $\underline{a}^{1,\gamma}$ for t''. Hence $\inf\{a^0, \underline{a}^{1,\gamma}(t')\} \leq \inf\{a^0, \underline{a}^{1,\gamma}(t'')\}$ and hence $\underline{a}^{2,\gamma}(t') \leq \underline{a}^{2,\gamma}(t'')$. By induction, $\underline{a}^{k,\gamma}$ obtained by iterating a^0 on $F(\cdot, t')$ is no greater than $\underline{a}^{k,\gamma}$ obtained by iterating a^0 on $F(\cdot, t'')$ for any k. Hence, $\liminf_k \underline{a}^{k,\gamma}(t') \leq \liminf_k \underline{a}^{k,\gamma}(t'')$ and hence $\underline{a}(t') \leq \underline{a}(t'')$. Proof of (ii). Now suppose a^0 is a fixed point of $F(\cdot, t')$. We show $a^0 \leq \underline{a}^{k,\gamma}(t'')$ for any k and γ. Observe that $a^0 \leq \overline{F}(a^0, t') \leq \underline{F}(a^0, t'') = \underline{a}^{1,\gamma}(t'')$. Clearly, $a^0 \leq \inf\{a^0, \underline{a}^{1,\gamma}(t'')\}$ and hence $a^0 \leq \underline{a}^{2,\gamma}(t'')$. By induction we obtain $a^0 \leq \underline{a}^{k,\gamma}(t'')$ for any k and γ and hence $a^0 \leq \liminf_k \underline{a}^{k,\gamma}(t'')$ and so $a^0 \leq \underline{a}(t'')$. $\square$

Remark 2. *We now show that the strong monotonicity of $F(a, \cdot)$ cannot be relaxed in (ii) of the Theorem 3. Let $T = \{0, 1\}$ and $A = [0, 1]$ with natural orders. Define*

$$F(a,0) := \begin{cases} \{0\} & if \quad a \leq \frac{1}{2} \\ \left[0, \frac{1}{2}\right] & if \quad a = \frac{1}{2} \\ \left\{\frac{1}{2}\right\} & if \quad a > \frac{1}{2}, \end{cases} \quad and \quad F(a,1) := \begin{cases} \left\{\frac{1}{4}\right\} & if \quad a \leq \frac{1}{2} \\ \left[\frac{1}{4}, 1\right] & if \quad a = \frac{1}{2} \\ \{1\} & if \quad a > \frac{1}{2}. \end{cases}$$

Obviously, this correspondence is not strongly monotone in t, but the other assumptions of Theorem 3 are satisfied. In particular, $a^0 = \frac{1}{2}$ is a fixed point of $F(\cdot, 0)$, but $\underline{a}(1) = \frac{1}{4}$ which violates (ii) in Theorem 3.

Observe that $a^0 = \frac{1}{2}$ is a common fixed point of $F(\cdot, 0)$ and $F(\cdot, 1)$. We can construct another, rather trivial, example in which a^0 is a fixed point of $F(\cdot, 0)$, but not $F(\cdot, 1)$. Let $A = [0, 1] \times [0, 1]$ with the product order, and $T = \{0, 1\}$ with the standard order. Let $F(a, 0) = [0, 1] \times [0, 1]$, and let $F(a, 1) = \left[\frac{1}{4}, 1\right] \times \left[\frac{1}{4}, 1\right]$ for all $a \in A$. Clearly, $a^0 = (0, 1)$ is a fixed point of $F(a, 0)$, but not $F(a, 1)$. In addition, $\underline{a}(1) = \left(\frac{1}{4}, \frac{1}{4}\right)$ which is incomparable with a^0.

5 Results for aggregate comparative statics

In this section, we provide results on monotone "aggregate" comparative statics that concern the comparative statics of some statistics or aggregates φ of endogenous variables. These results differ from most results in the existing literature, which typically concern endogenous variables themselves. However, in numerous situations statistics may unambiguously increase or decrease with a change in an exogenous parameter, although the endogenous variables may not be monotone. One such situation was described in the motivating Example 1, where the statistic φ was the aggregate (team) output and F was the joint best-reply mapping.

As before A is a sigma-complete lattice and T a poset, and we endow $A \times T$ with the product ordering. We will assume throughout the section that an aggregate $\varphi : A \times T \to \mathbb{R}$ that is continuous on A and monotone on $A \times T$.

The following results allow us to extend the iterative comparative statics results to the case in which the mapping or correspondence $F : A \times T \to A$ is monotone on A but not necessarily on T. This is the case of our motivating example. In what follows, we first built an intuition for our results considering a best-response dynamics, i.e. $\gamma = 1$ (under some weaker assumptions) and then generalize it to any γ (but require stronger assumptions).

Best response dynamics Consider sequences: $(\underline{a}^k)_{k=0}^{\infty}$ and $(\underline{b}^k)_{k=0}^{\infty}$, i.e. adaptive learning sequenced iterated from $a^0 = b^0 \in A$ on $\underline{F}(\cdot, t')$ and $\underline{F}(\cdot, t'')$, respectively, for $\gamma = 1$.

Theorem 4. *Let $t'' > t'$ be given. Suppose $\varphi(\bigwedge_{k \geq n} \underline{b}^k, t'') \geq \bigwedge_{k \geq n} \varphi(\underline{b}^k, t'')$ for any (sufficiently large) n, and the following condition is satisfied:*

$$\begin{aligned} &\textit{if } \varphi(a, t') \leq \varphi(b, t'') \textit{ for some } a, b \\ &\textit{then } \varphi(\underline{F}(a, t'), t') \leq \varphi(\underline{F}(b, t''), t''). \end{aligned} \tag{4}$$

Then $\varphi(\underline{a}(t'), t') \leq \varphi(\underline{b}(t''), t'')$.

Note that the hypotheses of Theorem 4, as well as the hypotheses of the other three theorems of this subsection, concern only specific values of t' and t'' as well as specific elements of the adaptive learning sequences $(\underline{a}^k)_{k=0}^{\infty}$ and $(\underline{b}^k)_{k=0}^{\infty}$. Condition $\varphi(\bigwedge_{k \geq n} \underline{b}^k, t'') \geq \bigwedge_{k \geq n} \varphi(\underline{b}^k, t'')$ is strong. However, as the example below the proof shows, the theorem need not hold true when this condition is violated, even though condition (4) is satisfied. One might also argue that the condition is difficult both for interpreting and verification, because it refers to, and requires computing the sequence of iterations $(\underline{b}^k)_{k=0}^{\infty}$. We stated the condition to pin down what we need for the proof. However, for many statistics φ even the stronger condition that $\varphi(\bigwedge B, t) = \bigwedge\{\varphi(b, t) : b \in B\}$ for any $t \in T$ and any countable $B \subset A$ is satisfied, and this stronger condition does not require computing any sequence. A similar comment applies to the other three theorems of this section.

Proof. Since φ is monotone on $A \times T$ (by our maintained assumption) and $a^0 = b^0$, we have:

$$\varphi(a^0, t') \leq \varphi(b^0, t'').$$

By condition (4) and observing that $\underline{a}^1 = \underline{F}(a^0, t')$ as well as $\underline{b}^1 = \underline{F}(b^0, t'')$ we obtain that

$$\varphi(\underline{a}^1, t') \leq \varphi(\underline{b}^1, t'').$$

By induction,

$$\varphi(\underline{a}^k, t') \leq \varphi(\underline{b}^k, t'').$$

Hence, $\bigwedge_{k\geq n} \varphi(\underline{a}^k, t') \leq \bigwedge_{k\geq n} \varphi(\underline{b}^k, t'')$ for all n. By definition $\bigwedge_{k\geq n} \underline{a}^k \leq \underline{a}^k$ for any $k \geq n$, so by monotonicity of φ, we have that $\varphi(\bigwedge_{k\geq n} \underline{a}^k, t') \leq \bigwedge_{k\geq n} \varphi(\underline{a}^k, t')$, and by assumption $\bigwedge_{k\geq n} \varphi(\underline{b}^k, t'') \leq \varphi(\bigwedge_{k\geq n} \underline{b}^k, t'')$. These three inequalities give

$$\varphi(\bigwedge_{k\geq n} \underline{a}^k, t') \leq \varphi(\bigwedge_{k\geq n} \underline{b}^k, t'').$$

Taking the limit for $n \to \infty$ and using the continuity of φ on A we conclude that

$$\varphi(\underline{a}(t'), t') \leq \varphi(\underline{b}(t''), t'').$$

Example 4. *Observe that we always have $\varphi(\bigwedge_{k\geq n} \underline{b}^k, t'') \leq \bigwedge_{k\geq n} \varphi(\underline{b}^k, t'')$, and so condition $\varphi(\bigwedge_{k\geq n} \underline{b}^k, t'') \geq \bigwedge_{k\geq n} \varphi(\underline{b}^k, t'')$ in fact implies $\varphi(\bigwedge_{k\geq n} \underline{b}^k, t'') = \bigwedge_{k\geq n} \varphi(\underline{b}^k, t'')$. To see that this condition is critical, let us reconsider the sequence of best response iterations (i.e. adaptive learning sequences with $\gamma = 1$) in our motivating example. Specifically, consider two sequences: $(a_t)_{t=0}^{\infty}$ and $(b_t)_{t=0}^{\infty}$, where $a^0 = b^0 = (x, x)$ for some number $x \in (0, 1]$. Then $a^k = (\frac{x}{t'}, xt')$ for any odd k and $a^k = a^0$ for any even k. Similarly, $b^k = (\frac{x}{t''}, xt'')$ for any odd k and $b^k = b^0$ for any even k.*

But now let the aggregate statistics be given by: $\varphi(x_1, x_2, t) = x_1 + x_2$. Assume $t'' > t' > 1$. Then we have: $\varphi(\bigwedge_{k\geq n} \underline{b}^k, t'') = (1 + \frac{1}{t''})x < 2x = \bigwedge_{k\geq n} \varphi(\underline{b}^k, t'')$, and *so the assumption of Theorem 4 is violated. Observe, in this case, the conclusion does not hold either. Indeed:*

$$\varphi(\underline{a}(t'), t') = (1 + \frac{1}{t'})x > (1 + \frac{1}{t''})x = \varphi(\underline{b}(t''), t''),$$

even though $\varphi(a^k, t') \leq \varphi(b^k, t'')$ *for any* k.

An analogous result to this in Theorem 4 holds true for iterations on $\overline{F}$ if we replace $\varphi(\bigwedge_{k \geq n} \underline{b}^k, t'') \geq \bigwedge_{k \geq n} \varphi(\underline{b}^k, t'')$ with $\varphi(\bigvee_{k \geq n} \overline{b}^k, t') \leq \bigvee_{k \geq n} \varphi(\overline{b}^k, t')$ and condition (4) by

$$\begin{aligned} &\text{if } \varphi(a, t') \leq \varphi(b, t'') \text{ for some } a, b \\ &\text{then } \varphi(\overline{F}(a, t'), t') \leq \varphi(\overline{F}(b, t''), t''). \end{aligned} \tag{5}$$

We then obtain: $\varphi(\overline{a}(t'), t') \leq \varphi(\overline{b}(t''), t'')$.

A similar result obtains for the iterations started from a^0, a fixed point of $F(\cdot, t')$.

Theorem 5. *Let* $t'' > t'$ *be given. Suppose that* $a^0 \in F(a^0, t')$*; in addition,* $\varphi(\bigwedge_{k \geq n} \underline{b}^k, t'') \geq \bigwedge_{k \geq n} \varphi(\underline{b}^k, t'')$ *for any (sufficiently large)* n*, and the following condition is satisfied:*

$$\begin{aligned} &\textit{if } \varphi(a, t') \leq \varphi(b, t'') \textit{ for some } a, b \\ &\textit{then } \varphi(\overline{F}(a, t'), t') \leq \varphi(\underline{F}(b, t''), t''). \end{aligned} \tag{6}$$

Then we have $\varphi(a^0, t') \leq \varphi(\underline{b}(t''), t'')$.

Proof. As φ is monotone on $A \times T$, we have:

$$\varphi(a^0, t') \leq \varphi(a^0, t''),$$

and hence

$$\varphi(\overline{a}^1, t') = \varphi(\overline{F}(a^0, t'), t') \leq \varphi(\underline{F}(a^0, t''), t'') = \varphi(\underline{b}^1, t'').$$

Proceeding by induction, we obtain:

$$\varphi(\overline{a}^k, t') \leq \varphi(\underline{b}^k, t'')$$

for any k. So

$$\varphi(\bigwedge_{k\geq n}\overline{a}^k,t')\leq\bigwedge_{k\geq n}\varphi(\overline{a}^k,t')\leq\bigwedge_{k\geq n}\varphi(\underline{b}^k,t'')\leq\varphi(\bigwedge_{k\geq n}\underline{b}^k,t''), \tag{7}$$

where the first inequality follows from the monotonicity of ϕ on $A\times T$, and the last inequality follows from our assumption. Now observing that $a^0\leq\overline{F}(a^0,t')$ we have that the sequence $(\overline{a}^k)_{k=0}^{\infty}$ iterated from a^0 on $\overline{F}(\cdot,t')$ is increasing. As a result we have $a^0\leq\lim_k\overline{a}^k$ and so by monotonicity of φ on $A\times T$ we have $\varphi(a^0,t')\leq\varphi(\lim_k\overline{a}^k,t')$. As a result, recalling (7), we have:

$$\varphi(a^0,t')\leq\varphi(\lim_k\overline{a}^k,t')\leq\varphi(\underline{b}(t''),t'').$$

□

One might ask if the strong monotonicity in condition (6) in Theorem 5 can be relaxed to weak monotonicity defined as follows:

$$\begin{aligned}&\text{if for some } a,b \text{ we have } \varphi(a,t')\leq\varphi(b,t'')\\ &\qquad\qquad\text{then } \varphi(\underline{F}(a,t'),t')\leq\varphi(\underline{F}(b,t''),t'').\end{aligned} \tag{8}$$

Then answer is no as the following example proves:

Example 5. *Let $A=[0,1]\times[0,1]$ and $T=\{0,1\}$ with the usual order, $\varphi(a_1,a_2,0)=\varphi(a_1,a_2,1)=a_1+a_2$. Put $F(a_1,a_2,0)=[0,1]\times[0,1]$ and $F(a_1,a_2,1)=\left[\frac{1}{4},1\right]\times\left[\frac{1}{4},1\right]$ for any $a\in A$. Then $a^0=(1,0)$ is a fixed point of $F(a,0)$; $\underline{b}^{k,1}(1)=\left(\frac{1}{4},\frac{1}{4}\right)$ for any $k\in\mathbb{N}$. But $\varphi(a^0,0)=1$ and $\varphi(\underline{b}(1),1)=\frac{1}{2}$, which violates the assertion of Theorem 5. Condition (6) is violated, because*

$$\varphi\left(\frac{1}{4},\frac{1}{4},1\right)=\frac{1}{2}<2=\varphi(1,1,0).$$

The condition (8) holds, however. Indeed, if $t'=0$ then for any $a'\in A$

$$\{\varphi(\tilde{a},t'):\tilde{a}\in F(a',t')\}=[0,2].$$

if $t''=1$ then for any $a''\in A$

$$\{\varphi(\tilde{a},t''):\tilde{a}\in F(a'',t'')\}=\left[\frac{1}{2},2\right].$$

Generalized adaptive dynamics We now generalize Theorem 4 and 5 to adaptive learning with arbitrary γ. For this reason consider two adaptive learning sequences $(\underline{a}^{k,\gamma})_{k=0}^{\infty}$, and $(\underline{b}^{k,\gamma})_{k=0}^{\infty}$ with $b^{0,\gamma} = a^{0,\gamma} = b^0 = a^0 \in A$, where

$$\underline{a}^{k+1,\gamma} = \underline{F}(\inf\{\underline{a}^{k,\gamma}, ..., \underline{a}^{k-\gamma+1,\gamma}\}, t')$$

and

$$\underline{b}^{k+1,\gamma} = \underline{F}(\inf\{\underline{b}^{k,\gamma}, ..., \underline{b}^{k-\gamma+1,\gamma}\}, t'').$$

We then have:

Theorem 6. *Let $t'' > t'$ be given. Suppose $\varphi(\bigwedge_{k\geq n} \underline{b}^{k,\gamma}, t'') \geq \bigwedge_{k\geq n} \varphi(\underline{b}^{k,\gamma}, t'')$ for any (sufficiently large) n and γ, and the following condition is satisfied:*

$$\text{if } \varphi(a, t') \leq \varphi(b, t'') \text{ and } \varphi(a', t') \leq \varphi(b', t'') \text{ for some } a, b, a', b'$$

$$\varphi\left(\underline{F}(a \wedge a', t'), t'\right) \leq \varphi\left(\underline{F}(b \wedge b', t''), t''\right). \tag{9}$$

Then we have $\varphi(\underline{a}(t'), t') \leq \varphi(\underline{b}(t''), t'')$.

Condition (9) is stronger than condition (4). The previous condition referred to a pair a and b of elements of A, and the present condition refers to two pairs: a and b and a' and b'. It may seem surprising that the condition refers to only two pairs, because γ can be any positive integer in the definition of general adaptive learning sequences. So, one would expect the present condition to refer to any finite set of pairs. It is possible to require less due to the following important property of general adaptive learning sequences.

Lemma 1. *Let $\gamma \geq 2$ and consider a sequence $(\underline{a}^{k,\gamma})_{k=0}^{\infty}$, where*

$$\underline{a}^{k+1,\gamma} = f(\bigwedge\{\underline{a}^{k,\gamma}, ..., \underline{a}^{k-\gamma+1,\gamma}\})$$

for some increasing function $f : A \to A$ and given $a^{0,\gamma} \in A$. Then:

1. *for any $n \geq 0$ and $j \in \{1, ..., \gamma-1\}$ we have $\underline{a}^{n\gamma+j,\gamma} = f(\bigwedge\{\underline{a}^{n\gamma,\gamma}, \underline{a}^{n\gamma+j-1,\gamma}\})$,*
2. *for any $n \geq 1$ we have $\underline{a}^{n\gamma,\gamma} = f(\bigwedge\{\underline{a}^{(n-1)\gamma,\gamma}, \underline{a}^{n\gamma-1,\gamma}\})$.*

We relegate its proof to Appendix, and we will proceed to proving Theorem 6.

Proof. Since φ is monotone on $A \times T$ we have:

$$\varphi(a^{0,\gamma}, t') \leq \varphi(b^{0,\gamma}, t'').$$

By condition (9), taking $a = a' = a^{0,\gamma}$ and $b = b' = a^{0,\gamma}$ we obtain:

$$\varphi(\underline{a}^{1,\gamma}, t') \leq \varphi(\underline{b}^{1,\gamma}, t'').$$

Applying condition (9) again we obtain:

$$\varphi(\underline{a}^{2,\gamma}, t') \leq \varphi(\underline{b}^{2,\gamma}, t'').$$

Continuing by induction, if $\varphi(\underline{a}^{k-j,\gamma}, t') \leq \varphi(\underline{b}^{k-j,\gamma}, t'')$ for some k and each $j \in \{0, 1, \ldots, k\}$ then $\varphi(\underline{a}^{k+1,\gamma}, t') \leq \varphi(\underline{b}^{k+1,\gamma}, t'')$. To see that, observe that by Lemma 1 $\underline{a}^{k+1,\gamma} = \underline{F}(\bigwedge\{\underline{a}^{k,\gamma}, \underline{a}^{0,\gamma}\}, t')$. Similarly, $\underline{b}^{k+1,\gamma} = \underline{F}(\bigwedge\{\underline{b}^{k,\gamma}, \underline{b}^{0,\gamma}\}, t'')$. Applying condition (9) we conclude that:

$$\varphi(\underline{a}^{k+1,\gamma}, t') \leq \varphi(\underline{b}^{k+1,\gamma}, t'').$$

As a result we obtain:

$$\varphi(\underline{a}^{k,\gamma}, t') \leq \varphi(\underline{b}^{k,\gamma}, t''),$$

for any k. The rest of the proof is analogous to the proof of Theorem 4. Indeed, $\bigwedge_{k \geq n} \underline{a}^{k,\gamma} \leq \underline{a}^{k,\gamma}$ for any $k \geq n$, and so by monotonicity of φ on $A \times T$ we conclude:

$$\varphi(\bigwedge_{k \geq n} \underline{a}^{k,\gamma}, t') \leq \bigwedge_{k \geq n} \varphi(\underline{a}^{k,\gamma}, t') \leq \bigwedge_{k \geq n} \varphi(\underline{b}^{k,\gamma}, t'') \leq \varphi(\bigwedge_{k \geq n} \underline{b}^{k,\gamma}, t''),$$

where the last inequality follows from assumption $\varphi(\bigwedge_k \underline{b}^{k,\gamma}, t'') \geq \bigwedge_k \varphi(\underline{b}^{k,\gamma}, t'')$. Taking the limit with n and γ and using the continuity of φ on A we conclude that

$$\varphi(\underline{a}(t'), t') \leq \varphi(\underline{b}(t''), t'').$$

□

An analogous result to that of Theorem 6 holds for iterations on $\overline{F}$ if we replace $\varphi(\bigwedge_{k\geq n} \underline{b}^{k,\gamma}, t') \geq \bigwedge_{k\geq n} \varphi(\underline{b}^{k,\gamma}, t')$ with $\varphi(\bigvee_{k\geq n} \overline{b}^{k,\gamma}, t') \leq \bigvee_{k\geq n} \varphi(\overline{b}^{k,\gamma}, t')$ and condition (9) with

$$\text{if } \varphi(a, t') \leq \varphi(b, t'') \text{ and } \varphi(a', t') \leq \varphi(b', t'') \text{ for some } a, b, a', b'$$

$$\varphi\left(\overline{F}(a \vee a', t'), t'\right) \leq \varphi\left(\overline{F}(b \vee b', t''), t''\right).$$

We then obtain: $\varphi(\overline{a}(t'), t') \leq \varphi(\overline{b}(t''), t'')$. For this analogous result, we need an analogous version of Lemma 1 for $(\overline{a}^{k,\gamma})_{k=0}^{\infty}$. We relegate its statement (Lemma 2) to Appendix.

A similar result obtains for the iterations started from a^0, a fixed point of $F(\cdot, t')$.

Theorem 7. *Let $t' < t''$. Suppose that $a^0 \in F(a^0, t')$; in addition, $\varphi(\bigwedge_{k\geq n} \underline{b}^{k,\gamma}, t'') \geq \bigwedge_{k\geq n} \varphi(\underline{b}^{k,\gamma}, t'')$ for any (sufficiently large) n and γ, and the following condition is satisfied:*

$$\text{if } \varphi(a, t') \leq \varphi(b, t'') \text{ and } \varphi(a', t') \leq \varphi(b', t'') \text{ for some } a, b, a', b' \text{ then}$$

$$\varphi\left(\overline{F}(a \vee a', t'), t'\right) \leq \varphi\left(\underline{F}(b \wedge b', t''), t''\right). \tag{10}$$

Then we have $\varphi(a^0, t') \leq \varphi(\underline{b}(t''), t'')$.

Proof. From monotonicity of φ on $A \times T$ we have:

$$\varphi(a^0, t') \leq \varphi(a^0, t'').$$

By condition (10), taking $a = a' = a^0$ and $b = b' = b^0$ we obtain:

$$\varphi(\overline{a}^{1,\gamma}, t') \leq \varphi(\underline{b}^{1,\gamma}, t'').$$

Applying condition (10) again we obtain:

$$\varphi(\overline{a}^{2,\gamma}, t') \leq \varphi(\underline{b}^{2,\gamma}, t'').$$

Continuing by induction, if $\varphi(\overline{a}^{k-j,\gamma}, t') \leq \varphi(\underline{b}^{k-j,\gamma}, t'')$ for some k and each $j \in \{0, 1, \ldots, k\}$ then $\varphi(\overline{a}^{k+1,\gamma}, t') \leq \varphi(\underline{b}^{k+1,\gamma}, t'')$. To see that, observe that by Lemma 2,

$\overline{a}^{k+1,\gamma} = \overline{F}(\bigvee\{\overline{a}^{k,\gamma}, \overline{a}^{0,\gamma}\}, t')$. Similarly, $\underline{b}^{k+1,\gamma} = \underline{F}(\bigwedge\{\underline{b}^{k,\gamma}, \underline{b}^{0,\gamma}\}, t'')$. Applying condition (10) we conclude that:

$$\varphi(\overline{a}^{k+1,\gamma}, t') \leq \varphi(\underline{b}^{k+1,\gamma}, t'').$$

Observe that $(\overline{a}^{k,\gamma})_{k=0}^{\infty}$ is increasing for all γ. Indeed, since $\overline{a}^{0,\gamma} = a^0 \leq \overline{F}(a^0, t') = \overline{a}^{1,\gamma}$ and $\overline{a}^{k+1,\gamma} = \overline{F}(\overline{a}^{k,\gamma}, t')$, $(\overline{a}^{k,\gamma})_{k=0}^{\infty}$ is increasing by the monotonicity of $\overline{F}$. Consequently:

$$\varphi(\bigvee_{k\geq n} \overline{a}^{k,\gamma}, t') \leq \bigwedge_{k\geq n} \varphi(\underline{b}^{k,\gamma}, t'') \leq \varphi(\bigwedge_{k\geq n} \underline{b}^{k,\gamma}, t''). \tag{11}$$

Thus, by the monotonicity and continuity of φ, we have $\varphi(a^0, t') \leq \varphi(\limsup_n \overline{a}^{n,\gamma}, t') = \varphi(\lim_n \overline{a}^{n,\gamma}, t')$. Combining this with inequality (11), and referring again to the continuity of φ, we get:

$$\varphi(a^0, t') \leq \varphi(\liminf_n \underline{b}^{n,\gamma}, t'').$$

Since this last inequality holds for all γ, we have that $\varphi(a^0, t') \leq \varphi(\underline{b}(t''), t'')$, as desired. $\square$

Recall the motivating example Recall that in the motivating example a best-reply mapping was: $f(a_1, a_2, t) = (\frac{a_2}{t}, ta_1)$ and the aggregate was: $\varphi(a_1, a_2, t) = 2\min\{ta_1, a_2\}$. Thus $\varphi(f(a, t), t) = 2\min\{a_2, ta_1\} = \varphi(a, t)$. So condition (4) holds. We also have that $\varphi(\bigwedge B, t) = \bigwedge\{\varphi(b, t) : b \in B\}$ for any $B \subset A$. Hence Theorem 4 can be applied to obtain the iterative comparative statics result from our motivating example for $\gamma = 1$. Since f is a mapping, then condition (6) is also satisfied and so Theorem 5 can also be applied to obtain the iterative comparative statics result from the motivating example.

Similar conclusions can be obtained for any other $\gamma > 1$. To see that it suffices to verify condition (9). First observe that:

$$\begin{aligned} \varphi(f(a \wedge a', t'), t') &= \varphi(a \wedge a', t') = \\ &= \varphi((\min\{a_1, a_1'\}, \min\{a_2, a_2'\}), t') = 2\min\{ta_1, ta_1', a_2, a_2'\}. \end{aligned}$$

Now, $\varphi(a,t') \leq \varphi(b,t'')$ and $\varphi(a',t') \leq \varphi(b',t'')$, i.e. $\min\{t'a_1, a_2\} \leq \min\{t''b_1, b_2\}$ and $\min\{t'a'_1, a'_2\} \leq \min\{t''b'_1, b'_2\}$, implies

$$\min\{t'a_1, ta'_1, a_2, a'_2\} \leq \min\{t''b_1, t''b'_1, b_2, b'_2\},$$

which means that

$$\varphi(f(a \wedge a', t'), t') \leq \varphi(f(b \wedge b', t''), t'').$$

As a result Theorem 6 can be applied to obtain the iterative comparative statics result from the motivating example.

Finally, observe that, although condition in Theorem 7 is not satisfied for arbitrary[18] a, a', b, b', it is satisfied for elements of the two sequences $\{\overline{a}^{k,\gamma}\}_{k=0}^{\infty}$ and $\{\underline{b}^{k,\gamma}\}_{k=0}^{\infty}$ to which condition (10) is applied in the proof of Theorem 7. Indeed, let $t' \geq 1$ and consider a fixed point $a^0 = (\frac{x}{t'}, x)$ (for some $x \in (0,1]$) with the aggregate output of $2x$. f is single valued hence the sequence $\{\overline{a}^{k,\gamma}\}_{k=0}^{\infty}$ is constant. The sequence $\{\underline{b}^{k,\gamma}\}_{k=0}^{\infty}$ starting from $b^0 = a^0$ is given by $\underline{b}^{1,\gamma} = (\frac{x}{t''}, \frac{xt''}{t'})$ and $\underline{b}^{2,\gamma} = (\frac{x}{t''}, x)$ which is a fixed point of $f(\cdot, t'')$ for $t'' > t'$. The value of the aggregate is: $\varphi(a^0, t') = 2x \leq 2x = \varphi(b^0, t'')$ and $\varphi(a^{1,\gamma}, t') = 2x \leq 2x = \varphi(\underline{b}^{1,\gamma}, t'')$ but also

$$\varphi(f(a^0 \vee \overline{a}^{1,\gamma}, t'), t') = \varphi(\overline{a}^{2,\gamma}, t') = 2x \leq 2x = \varphi(\underline{b}^{2,\gamma}, t'') = \varphi(f(b^0 \wedge \underline{b}^{1,\gamma}, t''), t'').$$

The conclusion of Theorem 7 is hence that $\varphi(a^0, t') = 2x \leq 2x = \varphi(\underline{b}(t''), t'')$ as stated in the motivating example. It is straightforward to show that $\varphi(\overline{b}(t''), t'') = 2x\frac{t''}{t'}$.

6 Applications and extensions

6.1 Social learning on networks

Our first application is to studying social learning on networks. DeGroot's model, in which agents take weighted averages of the opinions they observe, is a commonly

[18] Taking $a = (2,1), b = (2,4), a' = (0,1), b' = (2,0)$ we have that $\varphi(a,1) = 1 \leq 2 = \varphi(b,2)$ and $\varphi(a',1) = 0 \leq 0 = \varphi(b',2)$ but $\varphi(f(a \vee a',1),1) = 1 > 0 = \varphi(f(b \wedge b',2),2)$.

applied approach to studying social learning. Obviously, this very specific type of learning cannot well describe many real-life situations of interest. Cerreia-Vioglio et al. (2023) recently suggested a more general model, in which an opinion aggregator is a mapping that satisfies certain axioms. In their model of an economy of n agents, the opinion profile is represented by a vector $a \in A = [0,1]^n$, and learning is represented by an opinion aggregator $F : [0,1]^n \to [0,1]^n$ that is monotone with respect to coordinate-by-coordinate ordering on $A = [0,1]^n$.[19] We will illustrate our approach to comparative statics by applying it to the setting studied by Cerreia-Vioglio et al. (2023).

Our first result says that when agents assign higher weights to higher opinions, then all agents have higher opinions in the long-run. More specifically, suppose that S is the set of all possible weight vectors, that is, $S = \{s = (s_1, ..., s_n) \in [0,1]^n : s_1 + ... + s_n = 1\}$, where s_i, the i-th coordinate of any s, represents the weight assigned to the i-th highest opinion. The set $T = S^n$ of weighting profiles (different agents may have different weight vectors) is partially ordered by $\preccurlyeq$, the coordinate-by-coordinate first-order stochastic dominance. Now, we consider two opinion aggregators $F : A \times T \to A$, which are defined as follows: Each agent orders opinions in the input vector from the lowest to the highest, and her output opinion is the weighted average or the weighted median of the input opinions, where the agent uses her own weights.

Proposition 1. *If $t' \preccurlyeq t''$, then: $\underline{a}(t') \leq \underline{a}(t'')$ and $\overline{a}(t') \leq \overline{a}(t'')$; and if a^0 is a fixed point of $F(\cdot, t')$, then $a^0 \leq \underline{a}(t'')$.*

That is, if the agents shift their weights towards higher opinions, then the opinion of each of them increases in the long-run in response to this shift.

Note that this result holds true even though the opinions need not converge in the long-run. In addition, it follows from the proof that F can be defined in a variety of other ways, for example, one may use the correspondence F that takes values between the weighted average and the weighted median, more precisely,

[19] In addition to monotonicity, they impose two other axioms: normalization ($F(k, ..., k) = (k, ..., k)$ for all $k \in [0,1]$) and translation invariance ($F(a_1 + k, ..., a_n + k) = F(a_1, ..., a_n) + (k, ..., k)$ whenever it makes sense). They all are satisfied in our application.

$F(a) = \{b \in A : F^1(a) \wedge F^2(a) \leq b \leq F^1(a) \vee F^2(a)\}$, where F^1 is the weighted mean and F^2 is the weighted median.

Proof. We will check the key assumption of Theorem 3 that F is weakly increasing on $A \times T$. All other assumptions are obviously satisfied. Suppose that $a' \leq a''$ and $t' \preccurlyeq t''$. Then it follows directly from the definition of first-order stochastic dominance and the definition of F that $F(a', t') \leq F(a', t'')$. To complete the proof we must show that $F(a', t'') \leq F(a'', t'')$. We will show that $(a')^{(k)} \leq (a'')^{(k)}$, where $a^{(k)}$ denotes the k-th lowest coordinate of a, and this will complete the proof. Indeed, if the $k-1$ lowest coordinates of a' are the same as the $k-1$ lowest coordinates of a'', then $(a')^{(k)}$ is the lowest of the remaining $n-k+1$ coordinates of a', and $(a'')^{(k)}$ is the lowest of the corresponding $n-k+1$ coordinates of a''. Then, $(a')^{(k)} \leq (a'')^{(k)}$, because $a' \leq a''$ in the coordinate-by-coordinate ordering. Otherwise, $(a'')^{(l)}$ for an $l < k$, is the coordinate of a'' corresponding to one of the $n-k+1$ highest coordinates of a'. This coordinate of (a'') is higher than the coordinate of a' it corresponds to, and so $(a'')^{(l)} \geq (a')^{(k)}$. This completes the proof, because $(a'')^{(k)} \geq (a'')^{(l)}$ by definition. □

We finally illustrate by an example that our approach allows for comparative statics beyond of what is covered in our results. More specifically, one may be interested how different initial opinions affect the opinions in the long-run. So, consider a group of agents $N = \{1, 2, 3\}$ who share their opinions $a^0 \in [0, 1]^3$. Suppose that the weights assigned to the other agents are represented by the matrix

$$W = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}.$$

Note that this time agents assign weights by identity not according to the ranking of opinions, that is, the entry in column j and row i of the matrix represents the weight assigned by agent i to the opinion of agent j.

Consider the aggregation induced by the median. For example, $a^1 = F(a^0) = (0.6, 0.6, 0.6)$ for $a^0 = (0.6, 0.6, 0.6)$ and $a^1 = F(a^0) = (0.6, 0.4, 0.6)$ for $a^0 =$

$(0.8, 0.6, 0.4)$. The median aggregator satisfies the conditions required by of Cerreia-Vioglio et al. Actually, the aggregator was used as an example in their paper.

When $a^0 = (0.6, 0.6, 0.6)$, then $\underline{a} = \overline{a} = (0.6, 0.6, 0.6)$. When $a^0 = (0.8, 0.6, 0.4)$, then $\underline{a}^{k,1} = \overline{a}^{k,1} = (0.6, 0.4, 0.6)$ for odd k and $\underline{a}^{k,1} = \overline{a}^{k,1} = (0.6, 0.6, 0.4)$ for even k. Thus,

$$\liminf_{k=\infty} \underline{a}^{k,1} = (0.6, 0.4, 0.4) \text{ and } \limsup_{k=\infty} \overline{a}^{k,1} = (0.6, 0.6, 0.6).$$

For $\gamma \geq 2$, $\underline{a}^{1,\gamma} = \overline{a}^{1,\gamma} = (0.6, 0.4, 0.6)$; and for $k \geq 2$, $\underline{a}^{k,\gamma} = (0.4, 0.4, 0.4)$ and $\overline{a}^{k,\gamma} = (0.6, 0.6, 0.6)$. So,

$$\liminf_{k=\infty} \underline{a}^{k,\gamma} = (0.4, 0.4, 0.4) \text{ and } \limsup_{k=\infty} \overline{a}^{k,\gamma} = (0.6, 0.6, 0.6).$$

This yields $\underline{a} = (0.4, 0.4, 0.4)$ and $\overline{a} = (0.6, 0.6, 0.6)$.

Therefore we conclude that if the initial opinions change from $(0.6, 0.6, 0.6)$ to $(0.8, 0.6, 0.4)$, then the opinions of all agents go down in the long-run. Intuitively, the reason is that the opinions of agent 1 are less influential than the opinions of agent 3.

6.2 Network effects in a Bertrand competition

Our second application concerns firm specific network effects in oligopolistic markets with differentiated products. We refer the reader to Katz and Shapiro (1985) for a motivation and framework for studying networks effects in the Cournot industry.[20] In contrast to their paper we analyze industries described by Bertrand price competition.

We start with a simple algebraic illustration. Consider a 2-player Bertrand competition with differentiated products. Suppose the demand of firm i facing competitor j is given by: $d_i(p_i, p_j) = z_i - 0.5p_i + \delta_i p_j$. Assume $\delta_i \in (0, 1)$. Marginal costs are equal to c_1 and c_2. Profit function of each company is given by:

$$\pi_i(p_i, p_j) = (z_i - 0.5p_i + \delta_i p_j)(p_i - c_i).$$

[20] See also Amir et al. (2021) for a comparison of industry and firm-specific network effects.

The best responses of each company are given by:

$$BR_i(p_j) = z_i + \delta_i p_j + 0.5c_i$$

and the unique Nash equilibrium is (p_1^{NE}, p_2^{NE}), where:

$$p_i^{NE} = \frac{z_i}{1-\delta_1\delta_2} + \frac{\delta_i z_j}{1-\delta_1\delta_2} + 0.5\frac{\delta_i c_j + c_i}{1-\delta_1\delta_2}.$$

Computing the equilibrium outputs (q_1^{NE}, q_2^{NE}) we obtain:

$$q_i^{NE} = 0.5\frac{z_i}{1-\delta_1\delta_2} + 0.5\frac{\delta_i z_j}{1-\delta_1\delta_2} + 0.5\frac{c_i}{1-\delta_1\delta_2}[\delta_1\delta_2 - 0.5] + 0.25\frac{\delta_i c_j}{1-\delta_1\delta_2}.$$

For the moment, for simplicity, we set $c_i = c_j = 0$.

Following the literature that studies the industry viability in the presence of network externalities we now analyze quantity dynamic as a function of firm specific network effects. To do that let us assume that the market size parameters (z_i, z_j) in the demand function are increasing functions of own expected production levels (a_i, a_j). This can be a result of a demand side economies of scale driven, e.g., by the snob or bandwagon effects. We start from a simple example where $z_i := g_i(a_i)$ for some increasing function $g_i : \mathbb{R}_+ \to \mathbb{R}_+$. The Bertrand equilibrium production levels (for zero marginal costs) are now as follows:

$$\begin{cases} q_1^{NE} &= \frac{1}{2}\frac{g_1(a_1)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_1 g_2(a_2)}{1-\delta_1\delta_2} \\ q_2^{NE} &= \frac{1}{2}\frac{g_2(a_2)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_2 g_1(a_1)}{1-\delta_1\delta_2}. \end{cases}$$

The own network effect $g_1(a_1)$ increases directly Nash equilibrium output q_1^{NE}, while a competitor's network effect $(g_2(a_2))$ increases firm 1 Nash equilibrium output only indirectly via equilibrium prices in a Bertrand competition. We will call it a *spillover effect*. We impose the rational expectation equilibrium (REE) condition requiring that in REE $a_i = q_i^{NE}$, i.e. the expected and realized production levels coincide for both firms. Network / production size dynamics starting from given $a^0 = (a_1^0, a_2^0)$ is given by iterating on function $f(a_1^k, a_2^k) := \left(\frac{1}{2}\frac{g_1(a_1^k)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_1 g_2(a_2^k)}{1-\delta_1\delta_2}, \frac{1}{2}\frac{g_2(a_2^k)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_2 g_1(a_1^k)}{1-\delta_1\delta_2}\right)$. The REE are fixed points of f.

The results of our paper can be directly applied to conduct comparative statics of the firm specific network effects dynamics for any pair of monotone functions (g_1, g_2). For example:

Proposition 2. *For any initial a^0 production level, the long-run bounds: $\underline{a}$ and $\overline{a}$ of the adaptive learning sequences are increasing in parameters $(\delta_1, \delta_2) \in (0,1)^2$. That is, higher product substitutability leads to higher long-run network effects and hence production levels. If a^0 is the REE for (δ_1, δ_2) then for any higher parameters δ_1', δ_2' we have $a^0 \leq \underline{a}(\delta_1', \delta_2')$.*

The above proposition follows directly from Theorem 3 when we observe that f is monotone in a and δ_1, δ_2. It is also intuitive, the higher the parameters of product substitutability, the higher the spillover effects of the network effects and hence the higher the long-run realized production levels. If marginal costs (c_1, c_2) are non-zero we also conclude:

Proposition 3. *The long-run bounds: $\underline{a}$ and $\overline{a}$ are increasing in marginal costs c_1, c_2, provided $\delta_1\delta_2 \in (0.5, 1)$.*

The condition $\delta_1\delta_2 \in (0.5, 1)$ means that the spillover effects are high enough to assure that an increase in marginal costs e.g c_1 leads to an increase of equilibrium output q_i^{NE} of both firms $i = 1, 2$. This is a sufficient condition for long-run production bounds to be increasing in marginal costs.

We finish this example with a numerical illustration which shows that our long-run bounds are often computable.

Numerical example Suppose $g_1(a) = g_2(a) = 2(a - (a-1)(a-0.5)a)$ for $a \in Z = [0, 1.14]$ with $\delta_1 = 0.02, \delta_2 = 0.05$ and $c_1 = c_2 = 0$. These functions guarantee multiplicity of REE, which is a common feature of oligopoly models with network effects. Indeed, in our example there are 3 stable REE: $(0,0)$, $(0.04645, 1.0045)$ and $(1.0386, 1.0823)$. Dynamics of network externalities in this model has an intuitive interpretation. Let us start with Picard iterations ($\gamma = 1$). If both firms start with small levels of $a^0 = (a_1^0, a_2^0)$ such iterative dynamics will lead to stable REE $(0,0)$ and the industry will not be viable. If both start with large levels of (a_1^0, a_2^0) (e.g. above 0.5) the iterative dynamics will converge to a greatest REE. There is an intermediate case as well, if the first firm starts from a small but the second from a large level, e.g. $(0.4, 0.48)$, the dynamics converges to the assymetric equilibrium

$(0.04645, 1.0045)$ where one firm dominates the market while the competitor has a small market share. Industry viability depends also on the considered learning process, i.e. for the initial point $a^0 = (0.2, 0.48)$ although for $\gamma = 1$ the industry dynamics converges to $(0,0)$, for any $\gamma \geq 2$ the lower bound is $\underline{a} = (0,0)$ and the upper bound is $\overline{a} = (0.04645, 1.0045)$. The reason for this is a fact that $f(a^0)$ is not ordered with respect to a^0 and the sequence $(a^{k,\gamma})_{k=0}^{\infty}$ is not monotone.

Theorem 3 can be also applied to more general (than linear) demand and costs functions, in particular these that generate multiple Nash equilibria in the underlying Bertrand competition, provided the analyzed monotone correspondence F possesses the greatest and the least selections. Also, more general forms of network effects are allowed e.g. *direct* network externalities of firm j on a demand of firm i by including a_j as an additional argument of function g_i.

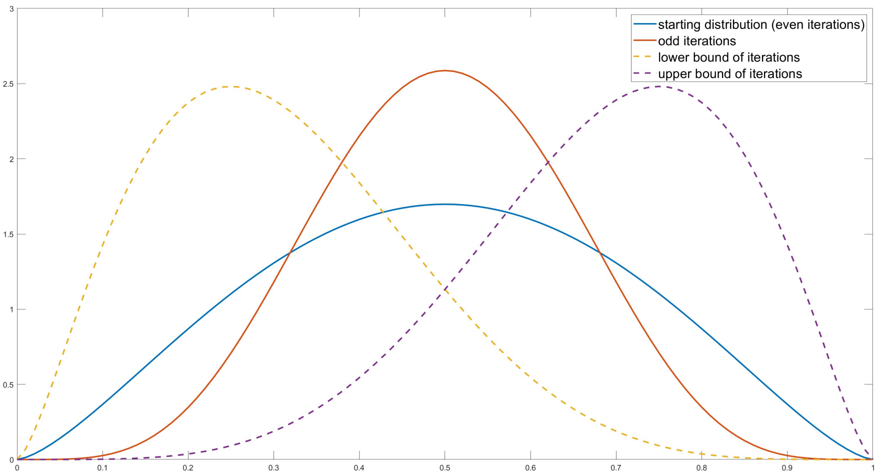
6.3 Distributional dynamics and long-run income distribution in monotone economies

We first provide an algebraic illustration of our techniques applied to a distributional comparative statics in the spirit of Camacho et al. (2018) (Section 6) and then apply our techniques to obtain general monotone comparative statics of the long-run income dynamics in monotone economies.

Distributional income dynamics Consider a continuum of agents of size 1, where each individual is characterized by its wealth or income. Normalize wealth / income on $[0,1]$ with a standard order. In this algebraic example, we model a distribution of wealth / income in the population using a family of beta distributions[21] $Be(x,y)$ with shape parameters (x,y). For example $Be(1,1)$ is the uniform distribution on $[0,1]$. Assume that $(x,y) \in [\epsilon, A-\epsilon]^2$ for some small $\epsilon > 0$, and large enough $A > \epsilon$. Suppose that processes in the economy (a summary of earnings, innovations, materialization of risks but also governmental policies like taxes

[21] A density of $Be(x,y)$ is $\rho(w;x,y) = \frac{1}{\xi(x,y)} w^{x-1}(1-w)^{y-1}\mathbf{1}_{[0,1]}(w)$, where $\xi(x,y)$ is the normalized constant. It is well known, $\xi(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, where $\Gamma(a) := \int_0^\infty w^{a-1}e^{-w}dw$ for every $a > 0$.

Figure 1: Cycles and long-run distribution dynamics for a family of beta distributions.



or subsidies, etc.) affect the wealth / income distribution via function f. In particular, let $X := \{Be(x, y) : (x, y) \in [\epsilon, A - \epsilon]^2\}$, and let $f : X \to X$ be defined, e.g., as follows $f(Be(x, y)) := Be(A - y, A - x)$. Endow X with the first order stochastic dominance. It is well known that the distribution $Be(x', y')$ stochastically dominates $Be(x, y)$ if $x' > x$ and $y > y'$. As a result, f is an increasing mapping. The set of fixed points of f is a set of invariant measures under this map that represent the stable income / wealth distribution. Such fixed points are given by: $\{Be(x, A - x) : x \in [\epsilon, A - \epsilon]\}$. This is totally ordered set with the least element $Be(\epsilon, A - \epsilon)$ and the greatest element $Be(A - \epsilon, \epsilon)$.

Now illustrate our results by studying distributional income / wealth dynamic $\{\mu^{k,\gamma}\}_{k=0}^{\infty}$ governed by f. For given x_0, let $\mu^0 := Be(x_0, x_0)$. Then for any odd k, $\mu^{k,1} = Be(A - x_0, A - x_0)$ and for even k, $\mu^{k,1} = Be(x_0, x_0)$. As a consequence, $\underline{\mu} = Be(\min(x_0, A - x_0), \max(x_0, A - x_0))$, and $\overline{\mu} = Be(\max(x_0, A - x_0), \min(x_0, A - x_0))$. In particular, for $x_0 < A/2$, $\underline{\mu} = Be(x_0, A - x_0)$ and $\overline{\mu} = Be(A - x_0, x_0)$, and for $x_0 > A/2$, $\overline{\mu} = Be(x_0, A - x_0)$ and $\underline{\mu} = Be(A - x_0, x_0)$.

For example, take $A = 8$ and $\mu = Be(2.5, 2.5)$. We get $\mu^{k,1} = Be(5.5, 5.5)$ for odd k, and $\mu^{k,1} = Be(2.5, 2.5)$ for even k. We get $\underline{\mu} = Be(2.5, 5.5)$ and $\overline{\mu} = Be(5.5, 2.5)$. Both are fixed points of f. Figure 1 illustrates the iterations.

Comparing long-run income distributions in monotone economies Our results can be applied to the comparative statics of long-run distributions of output or income associated with more abstract infinite horizon stochastic growth models with nonconvexities. Let the production or income available at period t be $y_t \in Y$, where $Y = [0, \bar{Y}] \subset \mathbb{R}_+$. Agent selects a consumption level $c_t \in [0, y_t]$, with the remaining resources $i_t = z_t - c_t$ allocated as an investment. The evolution of income is given by $y_{t+1} = f(i_t, z_{t+1})$ where f is a continuous, strictly increasing production function and z_{t+1} is a random shock drawn each period from distribution π over a finite set Z. For simplicity, we assume full depreciation. The temporal utility is given by a continuous, strictly increasing and strictly concave function $u : Y \to \mathbb{R}$. The agent's objective then is to maximize her expected discounted payoffs over an infinite horizon, given an initial state $y_0 \in Y$ and discount $\beta \in (0, 1)$. Denote the value of this optimization problem by $v^*(y_0)$. This problem admits a recursive representation, where $v = v^*$ is the unique solution to the Bellman equation:

$$v(z) = \max_{i \in [0,y]} u(y - i) + \beta \int_Z v(f(i, z'))d\pi(z').$$

Let the policy correspondence be given by

$$H^*(y, \beta) = \arg\max_{i \in [0,y]} u(y - i) + \beta \int_Z v(f(i, z'))d\pi(z').$$

Since u is strictly increasing and strictly concave, the objective has strictly increasing differences in $(i; y, \beta)$. Then, by an application of the Topkis (1998) Theorem (e.g., Theorem 2.8.4), the policy correspondence[22] $H^*(y, \beta)$ is a nonempty and jointly strongly increasing in (y, β).

Let $\mathcal{M}(Y)$ denote a set of measures on Y endowed with the first-order stochastic dominance and the weak convergence of measures. $\mathcal{M}(Y)$ is a complete lattice[23]. For a measurable set $B \subset Y$ define the stochastic income transition with $Q(B|i) := \int_Z 1_B(f(i, z'))d\pi(dz')$. For any selector $h_\beta(\cdot) \in H^*(\cdot, \beta)$, define the associated adjoint Markov operator:

$$\Lambda_{h_\beta}\mu(B) = \int_Y Q(B|h_\beta(y))\mu(dy)$$

[22] Recall, since f is not necessarily concave in i the policy is not necessarily unique.

[23] See, for example, Kamae et al. (1977)).

and the associated adjoint Markov correspondence:

$$\Lambda\mu(B) = \{\Lambda_{h_\beta}\mu(B)\}_{h_\beta \in H^*(\cdot,\beta)}.$$

Since H^* is strongly increasing, Λ is strongly increasing[24] on $M(Y)$.

We can now apply our Theorem 3 to characterize the iterative monotone comparative statics of the stationary income distributions.

Proposition 4. *When $\beta_1 \leq \beta_2$, for any initial measure $\mu_0 \in \mathcal{M}(Y)$, the lower (resp., upper) bounds for long-run income distribution dynamics are increasing, i.e. $\underline{\mu}(\beta_1) \leq \underline{\mu}(\beta_2)$ (resp., $\overline{\mu}(\beta_1) \leq \overline{\mu}(\beta_2)$), and from any stationary equilibrium at the discount rate β_1, say μ_{β_1}, we conclude that iterations on Λ_{β_2} from μ_{β_1} satisfy $\mu_{\beta_1} \leq \underline{\mu}(\beta_2)$.*

We remark that a similar reasoning can be applied to study the stationary equilibrium distribution in large dynamic economies in the spirit of Bewley or Huggett/Aiyagari models without aggregate risk. Indeed, interpreting μ as a distributions of income over Y in some large economy, we can study monotone comparative statics of stationary or invariant income distributions after the monotone exogenous shock to the policy function h in the income fluctuation problem of the shocks governed by Q for any initial income distribution μ^0.

In addition, our results extend the stationary equilibrium comparative statics for monotone economies based upon the work of Hopenhayn and Prescott (1992), Huggett (2003), and Acemoglu and Jensen (2015).

6.4 Comparing recursive equilibria in dynamic models with indeterminate equilibria

We finally show how to apply our results to obtain monotone comparative statics of (minimal state space) recursive equilibria (RE) in macroeconomic models with multiplicities.[25]

[24] See Huggett (2003) Theorem 1.

[25] See Coleman (1991) and Mirman et al. (2008) for motivation and other references.

Consider a simple stochastic OLG economy with production. There is a continuum of identical agents born each period who live for two periods. In the first period of life, they are endowed with a unit of time which they supply inelastically to the firm at the prevailing wage $w(s)$, and they consume and save. In the second period of life, they consume their savings which are subjected to a stochastic return $r(s')$. Here s and s' denote vectors of aggregate state variables in the current and the following periods. Preferences are time separable with discount rate $\beta \in (0,1)$ and are given by $u(c_1) + \beta v(c_2)$, where consumption when young (resp., old) is denoted by c_1 (resp., c_2), and $u : \mathbb{R}_+ \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ are smooth, strictly increasing, strictly concave, with $\lim_{c \to 0^+} u'(c) = \infty = \lim_{c \to 0^+} v'(c)$.

The reduced-form technology is given by $f(k,n)e(K,N,z)$ where f is a technology transforming private inputs of capital and labor (k,n), and the externality $e(K,N,z)$ is a total factor productivity that depends on per capital aggregates of capital and labor (K,N) and a shock $z \in Z = [z_l, z_h] \subset \mathbb{R}_{++}$ which is drawn each period from a first-order Markov process with stationary transition $\pi(z,z')$ that satisfies a Feller property. We let f satisfy typical assumptions, namely: it is constant returns to scale, increasing (but increasing strictly with each argument for the positive input of the other), weakly concave jointly (but strictly concave with each argument separately for the positive input of the other) and twice continuously differentiable. Moreover $r(k,z) := f_1(k,1)e(K,1,z)$ is decreasing in k and increasing in K for $K > 0$,[26] $\lim_{k \to 0^+} r(k,z) = \infty$, $\lim_{k \to 0^+} r(k, z_{\max})k = 0$; $w(k,z) := f_2(k,1)e(k,1,z)$ is increasing in k with $\lim_{k \to 0^+} w(k,z) = 0$; both r and w are increasing in z for all k. Finally, $f(0,1)e(0,1,z) = 0$ for any z and there exists a maximal sustainable capital stock[27] denoted by $k_{\max}$. Many examples of technologies that satisfy these assumptions can be given (see, e.g., Datta et al. (2018)). Further, under this assumption, we can restrict attention to compact state spaces for capital $X \subset \mathbb{R}_+$.

[26] This model is known to have multiple equilibria. Recall $f_1(k,1)e(K,1,z)$ under our assumptions is *mixed monotone* in (k,K), i.e., decreasing in k and increasing in K. This is a critical feature that creates the possibility for equilibrium indeterminacy (i.e. a continuum of equilibria) in this class of models (see Santos (2002) and Datta et al. (2018) for discussion).

[27] $\forall k \geq k_{\max}$ and $\forall z \in Z$, $F(k,1,k,1,z) \leq k_{\max}$.

Anticipating that $n = 1 = N$ and $k = K$ in any RE, and denoting the aggregate vector of state variables by $s = (K, z) \in S = X \times Z$, we consider the existence of RE in a class of investment functions W with pointwise partial orders. In particular, let $W = \{h : S \to \mathbb{R}_+,\ 0 \le h \le w,\ h(k, z)$ increasing in k, and measurable$\}$.

Together with π, $h \in W$ describes the law of motion for the aggregate variables. Taking this, a young agent solves:

$$\max_{y \in [0, w(s)]} u(w(s) - y) + \beta \int_Z v(r(h(s), z')y)\pi(z, dz'),$$

Let $\hat{y}(s; h)$ be the optimal solution to this household problem for $h \in W$. The optimal solution is unique under our assumptions.

Labor and capital markets are competitive hence by profit maximization, so in a RE, $w(K, z) = f_2(K, 1)e(K, 1, z)$ and $r(K, z) = f_1(K, 1)e(K, 1, z)$. A RE of this economy in the space W is a law of motion $h^* \in W$ and policy function $y^* \in W$ such that $y^*(s) = \hat{y}(s; h^*) = h^*(s) \in W$ for each $s \in S_{++}$ whenever $h^*(s) > 0$. Here $S_{++} := X_{++} \times Z$ with $X_{++} \subset \mathbb{R}_{++}$. Market clearing is implied by the formulation of the household problem.

Now, consider the question of capital deepening in the discount rate β to the set of RE in this economy. Define a nonlinear operator F_β on W as follows: for $h(s) > 0$, $h \in W$, let $F_\beta h(s)$ be the unique y solving:

$$u'(w(s) - y) - \beta \int_Z v'(f_1(h(s), 1)e(y, 1, z')y) f_1(y, 1)e(h(s), 1, z')\pi(z, dz') = 0, \quad (12)$$

with $F_\beta h(s) = 0$, whenever $h(s) = 0$ in any state s. Therefore, any function $h^*_\beta \in W$ is an RE law of motion if and only if it is a non-zero fixed point of the operator F_β in W. It is easy to prove F_β is a monotone operator on W. Further, under our assumptions, it can be show there exists a continuous $h_0 \in W$ such that $\forall h \ge h_0 > 0$, $F_\beta h > h_0$ on S_{++}.[28] Let $A = W \cap [h_0, w]$, and endow A with its relative pointwise partial order $\le$.

Observe that Theorem 3 does not guarantee existence of RE. To see that the set of RE is non-empty in the analyzed economy observe that $(A, \le)$ is a sigma-

[28] See McGovern et al. (2013) Proposition 2. In fact, $h_0 \in W$ is continuous jointly in (k, z).

complete lattice and $F_\beta : A \to A$ is an order-continuous self map, hence, there exists a RE by an application of a Tarski-Kantorovich fixed point theorem.[29]

Applying Theorem 3 to the operator F_β we obtain:

Proposition 5. *For any initial $h^0 \in A$ and $\beta_1 \le \beta_2$, we have: (i) the lower bounds for RE satisfy $\underline{h}(\beta_1) \le \underline{h}(\beta_2)$ and the upper bounds satisfy $\overline{h}(\beta_1) \le \overline{h}(\beta_2)$, (ii) taking initially any RE $h^*_{\beta_1} \in A$ but iterating on F_{β_2} we also have $h^*_{\beta_1} \le \underline{h}(\beta_2)$.*

A few remarks about the Proposition. First, the interpretation of proposition is immediate: even in the presence of a possible continuum of RE in $(A, \le)$, from any initial RE $h^*_{\beta_1} \in A$ of the less "patient" economy, the iterative process of computing the RE for the more patient economy is bounded below by $h^*_{\beta_1} \in A$.

Second, in dynamic stochastic equilibrium models with uncountable shocks the analyzed function space A (with the pointwise partial orders) is only sigma-complete. Hence, none of the results of Echenique (2002, 2004) can be applied. Yet, based on Theorem 3, we obtain a result on iterative monotone comparative statics of (monotone and measurable) RE. If the shocks are discrete, then the space $(A, \le)$ is a complete lattice and by Tarski's theorem the set of RE on $(A, \le)$ is a nonempty complete lattice for each β. The novelty of applying Theorem 3 in such settings is that it provides a monotone comparative statics of the long run (adaptive learning) bounds, and not only the comparative statics of extremal RE.

Applications of Theorem 3 can also be proposed in macroeconomic models with infinitely lived agents as well as with equilibrium indeterminacies (see for example Benhabib and Farmer (1994) and Datta et al. (2018)). Even in nonconvex, nonoptimal dynamic economies, where the optimal household decisions are correspondences, Theorem 3 can be applied to obtain similar iterative monotone comparative statics result as these correspondences satisfy conditions of Theorem 3 under standard assumptions.[30]

[29] It is also known that RE also exist (see Morand and Reffett (2007)) in subsets of W, where the elements of these subsets are additionally (i) lower or (ii) upper semicontinuous in k.

[30] See Mirman et al. (2008) for examples of such economies and assumptions guaranteeing existence of RE.

A Appendix

Proof to Theorem 2. We will show that $\underline{a} \geq a^*$; the proof that $\overline{a} \leq a^*$ is analogous. Let $a \ll a^*$. Since $(\underline{a}^{k,\gamma})_{k=0}^{\infty}$ is an adaptive learning sequence, $\underline{a}^{k,\gamma} > a$ for large enough values of k. Therefore, $\bigwedge\limits_{l \geq k} \underline{a}^{l,\gamma} \geq a$ for large enough values of k, what implies that $\liminf_k \underline{a}^{k,\gamma} \geq a$. This is true for all γ. Since $\underline{a}$ is the limit of the decreasing sequence $(\liminf_k \underline{a}^{k,\gamma})_{\gamma=1}^{\infty}$, we obtain that $\underline{a} \geq a$. Since this is true for all $a \ll a^*$, we have that $\underline{a} \geq \sup\{a \in A : a \ll a^*\}$. By condition (3), we obtain that $\underline{a} \geq a^*$.

Finally, $\underline{a} \geq a^*$, $\overline{a} \leq a^*$ and $\underline{a} \leq \overline{a}$ deliver the result.

Proof to Lemma 1 By definition, for $1 \leq k \leq \gamma$ we have $\underline{a}^{k,\gamma} = f(\bigwedge\{\underline{a}^{j,\gamma}, j = 0, 1, \dots, k-1\})$. Since

$$\bigwedge\{\underline{a}^{j,\gamma}, j = 0, 1, \dots, k-1\} \leq \bigwedge\{\underline{a}^{j,\gamma}, j = 0, 1, \dots, k-2\},$$

we have: $\underline{a}^{k,\gamma} = f(\bigwedge\{\underline{a}^{j,\gamma}, j = 0, 1, \dots, k-1\}) \leq f(\bigwedge\{\underline{a}^{j,\gamma}, j = 0, 1, \dots, k-2\}) = \underline{a}^{k-1,\gamma}$, for $k \geq 2$, by the monotonicity of f. Thus the sequence $(\underline{a}^{k,\gamma})_{k=1}^{\gamma}$ is decreasing. As a consequence:

$$\underline{a}^{k,\gamma} = f(\bigwedge\{\underline{a}^{0,\gamma}, \underline{a}^{k-1,\gamma}\})$$

for any $1 \leq k \leq \gamma$. So thesis 1 is true for $n = 0$ and thesis 2 for $n = 1$. Next, since the sequence $(\underline{a}^{k,\gamma})_{k=1}^{\gamma}$ is decreasing:

$$\underline{a}^{\gamma+1,\gamma} = f(\bigwedge\{\underline{a}^{\gamma,\gamma}, ..., \underline{a}^{1,\gamma}\}) = f(\underline{a}^{\gamma,\gamma}).$$

Since

$$\bigwedge\{\underline{a}^{\gamma+j,\gamma}, j = 0, 1, \dots, k-1\} \leq \bigwedge\{\underline{a}^{\gamma+j,\gamma}, j = 0, 1, \dots, k-2\}, \tag{13}$$

we have, by the monotonicity of f, that $\underline{a}^{\gamma+k,\gamma} = f(\bigwedge\{\underline{a}^{\gamma+j,\gamma}, j = 0, 1, \dots, k-1\}) \leq f(\bigwedge\{\underline{a}^{\gamma+j,\gamma}, j = 0, 1, \dots, k-2\}) = \underline{a}^{\gamma+k-1,\gamma}$ for any $k = 2, \dots, \gamma$. As a consequence:

$$\underline{a}^{\gamma+k,\gamma} = f(\bigwedge\{\underline{a}^{\gamma,\gamma}, \underline{a}^{\gamma+k-1,\gamma}\}).$$

So thesis 1 is true for $n = 1$ and thesis 2 for $n = 2$. Continuing in this manner we obtain:

$$\underline{a}^{n\gamma+k,\gamma} = f(\bigwedge\{\underline{a}^{n\gamma,\gamma}, \underline{a}^{n\gamma+k-1,\gamma}\}),$$

for arbitrary n and $k = 2, \ldots, \gamma$ and

$$\underline{a}^{n\gamma+1,\gamma} = f(\underline{a}^{n\gamma,\gamma}).$$

This completes the proof.

Lemma 2. *Let $\gamma \geq 2$ and consider the sequence $(\overline{a}^{k,\gamma})_{k=0}^{\infty}$, where*

$$\overline{a}^{k+1,\gamma} = f(\bigvee\{\overline{a}^{k,\gamma}, ..., \overline{a}^{k-\gamma+1,\gamma}\})$$

for some increasing function $f : A \to A$ and given $a^{0,\gamma} \in A$. Then:

1. *for any $n \geq 0$ and $j \in \{1, ..., \gamma-1\}$ we have $\overline{a}^{n\gamma+j,\gamma} = f(\bigvee\{\overline{a}^{n\gamma,\gamma}, \overline{a}^{n\gamma+j-1,\gamma}\})$,*
2. *for any $n \geq 1$ we have $\overline{a}^{n\gamma,\gamma} = f(\bigvee\{\overline{a}^{(n-1)\gamma,\gamma}, \overline{a}^{n\gamma-1,\gamma}\})$.*

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